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APPLICATION OF THE LORENTZ TRANSFORMATION TO THE GENERAL SECOND ORDER SOLUTION OF THE PROBLEM OF ETHER DRIFT MEASUREMENT

by

Henry Lisman

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(A.M., Boston University, 1935)

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APPROVED

BY

First Reader Royal M. Frye
ass't Professor of Physics

Second Reader Norton A. Kent.
Professor of Physics

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INTRODUCTION

The theory that light consists of a wave motion in a luminiferous ether was once generally accepted by physicists. According to this theory the ether is supposed to possess the property whereby waves of light may be transmitted and all other optical phenomena may be accounted for. It is assumed that all space, even that occupied by material bodies, is pervaded by this ether, and yet, it permits all bodies to move through it with free motion.

The "ether drift" experiment was originally conceived as an experimental means of ascertaining the actual validity of this hypothesis. The originator of this idea was James Clerk Maxwell in his article on "Ether" in the 1878 edition of the Encyclopedia Britanica. In this article he pointed out that, on the assumption that there is a stationary ether through which the earth is moving freely with an absolute velocity and that light waves are propagated in this free ether in any direction and always with the same velocity relative to the ether, the apparent velocity of light would depend upon whether the light is travelling along the direction of the motion of the earth or at right angles to it. The former path would be slightly lengthened as compared with the latter. Thus a relative motion between the earth and the stationary ether, that is, an "ether drift" should be detectable.

Since the velocity of the earth in its orbit is thirty kilometers per second and the velocity of light is ten thousand times as great, the velocity measured in the line of motion would differ from the velocity at

right angles to this line by thirty kilometers per second, that is, by one part in ten thousand. This would be a "first order effect". However, since all experimental methods of determining this effect require that the light shall travel from one point to another and back again, a positive effect of the earth's motion on the ray in the outgoing path would be nullified by the opposite effect on the returning ray, except for a "second order effect" due to the motion of the observer during the time of passage of the light. Concerning this effect, Maxwell concluded that "The change in the time of transmission of the light on account of a relative velocity of the ether equal to that of the earth in its orbit would be only one hundred-millionth part of the whole time of transmission and would therefore be quite insensible".

In 1880, while attending the University of Berlin, the late Professor Albert A. Michelson invented the interferometer which bears his name and is generally considered as being especially suited for detecting the hypothecated ether drift-- if it exist. By means of this apparatus, light incident upon a half-silvered plane parallel glass plate, is divided into two beams so that each travels along a separate path in a different direction. After having traversed their respective paths, each ray is reflected back again to the half-silvered mirror where they are again united and sent out in directions determined by the inclinations of the two mirrors. These inclinations are such that the final rays are almost parallel. If the two paths are optically equal in length, the reunited beams of light will blend and there will be no interference. If however, one path is longer than the other, there will result differences of phase which will give rise to interference fringes. As the interferometer is rotated with reference to the earth , first one arm has a relatively longer path and than the other; so that the interference fringes

which are actually due to the phase relationships between the two wave fronts of the final reunited rays will undergo slight shifts. The relation is such that an effective change of one wave length in the length of one path as compared with the other will result in a movement of the fringes equal to one fringe width.

The first experiment with this apparatus was carried out by A. A. Michelson in 1881. Later, Michelson together with Professor Edward W. Morley of Western Reserve University, as well as several other scientists, have at various times constructed interferometers and conducted experiments for the detection of this expected shift of fringes. In the July 1933 issue of the Reviews of Modern Physics Professor Dayton C. Miller presents a summary of the results of these investigators together with a very thorough analysis of extensive investigations of his own. The conclusion at which he arrived is that there exists a small, actual, periodic fringe shift which however is much too small to be accounted for by the hypothesis of a stationary ether.

The Michelson-Morley experiment with its null result suggested that either the existing theory of the ether was incorrect or that it was at least incomplete.

In order to account for the absence of any effect of the earth's translation, Prof. G. F. FitzGerald in 1891 put forth a theory which was put on a firm mathematical basis by H.A. Lorentz in 1895. According to this theory, the dimensions of a solid body undergo slight changes of the order of $(v/c)^2$ when it moves through the ether. The negative result of the experiment is thus accounted for by this contraction.

Following the contraction theory, the next theory introduced to

account for the null effect of the "ether drift" experiment, came in 1905 with Professor Albert Einstein's Special Theory of Relativity according to which a material body suffers a contraction in the direction of its motion. Upon the basis of this theory the "ether drift" experiment should have a null result. However, because of the definite positive effect obtained by the very extensive and thorough work of D. C. Miller, even though this effect is much smaller than that originally predicted, an hypothesis which leads to a null effect is no more gratifying than the original stationary ether hypothesis. Nevertheless, in examining the broader postulates[1] of Professor Einstein's General Theory of Relativity announced in 1915 it will be observed that they may be modified so as not exclude the possibility of a relative velocity of drift. Indeed such triumphs of the General Theory as: the prediction of the bending of a ray of light when passing through a strong gravitational field, accounting for the advance of the perihelion of mercury and predicting that when the source of light is in the atmosphere of a massive star the wave length becomes slightly greater; are of such significance that they have been widely accepted as confirming the General Theory. On the other hand, the Special Theory with its prediction of a null effect may not actually be in violation of the experimental results of the ether drift experiment since it is a special case of the General Theory. The investigation of Michelson and Morley and also the later and more extensive work of D. C. Miller were carried out by means of equal arm interferometers in which the two beams of light traveled along paths which were at right angles to each other.

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1. Eddington, A. S., "The Mathematical Theory of Relativity" p. 104, (1930)
Cambridge University Press

At a meeting of the American Physical Society held at Wellesley College in the fall of 1937, W. B. Cartmel presented a paper in which he claimed that for a maximum fringe shift the arms should be set at an angle of forty five degrees. With this in view, E. E. Haskins carried out an investigation in which the arms were assumed to be set at this forty five degree angle. He arrived at the conclusion that the fringe shift was even greater than that predicted by Cartmel's formula. In fact he found, that in least one respect Cartmel's formula was in error.

The foregoing remarks are intended to make clear the necessity of carrying out a careful investigation of the entire problem with the view of determining the exact effect to be expected.

In this paper an investigation is carried out in which the lengths of the paths traversed by the two beams are made unequal. Also, the angle between the two arms is assumed to be arbitrary so that the final solution will be entirely general and therefore applicable to the conditions given in any particular experiment.

All of the equations involved in the problem will be set up from the point of view of an observer fixed in the ether. To make the results applicable, the Lorentz transformations will be applied so that the final result will come out in terms of the observer moving with the earth.

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REVIEW OF THE WORK OF OTHERS

The history of the "Ether-Drift" experiment had its inception when A. A. Michelson, while yet a student in Germany conducted experiments both in Berlin and in Potsdam in the years 1880-1881. These were the first attempts made to detect the second-order effect of the relative motion of the earth and the ether. The original attempts of Michelson[1] were unsuccessful. He observed approximately one-hundredth of the effect for which he was looking. This null effect however, was attributable, at least partially, to the fact that the light path was only a little more than one meter in length.

In 1881 Michelson was appointed to the professorship of physics in the Case School of Applied Science in Cleveland where he became acquainted with Professor Edward W. Morley, professor of chemistry in the neighboring Western Reserve University. With new developments in the interferometer and in the method of using it, Michelson, together with Morley, again attempted to detect the then expected effect. In order to avoid disturbance of vibration and distortion, the optical parts of the instrument were mounted on a solid block of sandstone which was floated in mercury contained in a circular tank of cast iron. Thus, it became possible to turn the interferometer to different azimuths while observations were in progress. To obtain the required sensitivity, the effective light path was increased by reflecting the light back and forth so that it traversed the diagonal of the square stone block

1. Michelson A. A., Am. J. Sci. (3) 22, 120 (1880)

eight times, giving the effect of an interferometer with an arm of approximately 1100 centimeters in length. The series of observations made by Michelson and Morley were of six hours duration and were taken during six different intervals of one hour each. The series of observations consisted of thirty six turns of the interferometer, sixteen readings being made at equidistant points of each turn. This relatively short series of observations showed that the effect was neither of the expected magnitude nor of zero value. The conclusions of this experiment[1], published in 1887 stated that the observed relative motion of the earth and the ether was not greater than one-fourth of the earth's orbital velocity.

In an address delivered in 1900 at the International Congress of Physics held in conjunction with the International Exposition in Paris, Lord Kelvin explained the significance of the Michelson-Morley experiments as related to the theories of the ether and he expressed the belief that further, more refined experimental work in this field was necessary. Accordingly E. W. Morley and D. C. Miller, who were present at this conference, constructed an interferometer having a light path more than three times as long as that used by Michelson and Morley in 1887. Originally, they employed a framework of heavy lumber but this was soon abandoned in favor of one made of structural steel and floated in a tank of mercury.

In the years 1900 to 1907[2] numerous observations were made; first, in the basement of the Case School of Applied Science, and later, on Euclid

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1. Michelson A. A. and Morley E. W., Am. J. Sci. (3) 34, 333 (1887).
2. Morley E. W. and Miller D. C., Phil. Mag. (6) 9, 680 (1905).

Heights, Cleveland. This latter site is at an altitude of 285 meters. The results obtained showed a very definite positive effect slightly larger than that previously obtained but still too small to be reconciled with the expected effect. At this time Professor Morley retired from active work and the work was continued by D. C. Miller in 1921.

Through the courtesy of the Carnegie Institute the ether-drift interferometer was set up on the grounds of the Mount Wilson Observatory in March 1921. Later the apparatus was returned to Cleveland where numerous observations and improvements were made during the years 1922 and 1923. In 1924, it was again returned to Mount Wilson where there were employed many refinements such as : improved mirror mountings, protection from heat, improved light source, large viewing telescope and other refinements which were developed in the laboratory tests at Cleveland.

Over two hundred thousand readings were taken with this instrument. This represents the most exhaustive study in this field. The analysis of the results of this work, together with a rather lengthy review of the whole topic in general and his own work in particular,are given by D. C. Miller in the Review of Modern Physics[1]. Miller reaches the conclusion that there does exist a small positive fringe shift equal to about five per cent of the expected effect.

Among other investigators who have studied this problem, we find R. J. Kennedy and E. M. Thorndike[2] who carried out experiments with an interferometer of a different design. The arms of their apparatus were made

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1. Miller D. C. Rev. Mod. Phys.,5, 203,(1933).

2. Kennedy R. J. and Thorndike E. M. Phys. Rev., 42, 400, (1932).

unequal in length and were set at such an angle to each other that the light reflected from the plate would be plane polarized. This angle they found to be sixty-five degrees. However, it does not seem that the investigators expected that the effect would be any greater by choosing that angle. The length of the light path in their apparatus was relatively short and they reported an extremely small effect, if any. In their report, they develop a theory based on the Lorentz-FitzGerald contraction, from which they conclude that no effect should be expected, regardless of the magnitude of the angle between the arms.

In 1926[1] and 1927[2], A. Piccard and E. Stahel reported results of experiments automatically performed and recorded in balloons at an altitude of 1900 meters. According to these experiments, there was a positive effect in rather close agreement with the results of D. C. Miller.

Professor Georg Joos[3], in Jena, working with an interferometer, mounted on a quartz base suspended in an evacuated metal housing and provided with photographic registration obtained results indicating that any existing ether drift could not exceed one kilometer per second.

The theoretical phase of the ether drift problem has been treated by a few investigators. In a paper by W. M. Hicks[4] of University College, Sheffield, there is an important theoretical investigation of the original experiment. In his work, Hicks considered, in particular, the effect in an

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[1] Piccard, A. and Stahel, E., Comptes Rendus 183, 420, (1926).

[2] Piccard, A. and Stahel, E., Comptes Rendus 185, 1198, (1927).

[3] Joos, Georg, Ann. D. Physik (5), 7, 385, (1930).

[4] Hicks, W. M., Phil. Mag., 3, 9, (1902).

interferometer of which the arms are at right angles to each other and the normals to the mirrors make a slight angle with the arms. From his investigation he concluded that this slight inclination introduces a small first order effect which is singly periodic in a complete rotation of the interferometer. The amplitude of this full period effect varies inversely as the width of the fringes being used at the time of the observations. Miller states that this effect was present in all the observations including the original observations of Michelson and Morley.

H. A. Lorentz in his book "The Theory of the Electron"[1] develops a theory that a body contracts in the direction of its motion. He discusses the ether drift experiment and derives the well known result $2\frac{v^2}{c^2}$.

The theory of the ether drift experiment was also discussed by A. Righi[2,3]. He concluded that there should be no fringe shift detectable, because, on account of the rotation, the two beams exchange places and hence the effect cancels out.

E. R. Hedrick[4] and Paul S. Epstein[5] presented papers dealing mainly with the effect produced when the mirrors are slightly oblique to the arms. In his discussion of the ninety degree interferometer, Epstein states that the angle between the rays in their final path is $\frac{v^2}{c^2}\cos 2\phi$. The latter expression corresponds to equation 58 in this paper.

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1. Lorentz H. A., "The Theory of the Electron", B. G. Teubner, Leipzig, (1916)
2. Righi A. , Comptes Rendus, 168, 837, (1919)
3. Righi A. , Comptes Rendus, 170, 499 and 1550, (1920)
4. Hedrick E. R., Astrophys. J., 68, 374, (1928)
5. Epstein P. S., Astrophys. J., 68, 383, (1928)

The "Ether Drift Conference"[1] gives evidence of the amount of attention that scientists have given this problem. This conference was held on February 4 and 5, 1927, at the Mount Wilson Obserbatory in Pasadena, California. The attention of the entire conference was centered upon the problem of measuring the ether drift. Among those present at this conference were Michelson, Lorentz, Miller, Kennedy, Hedrick and Epstein.

W. B. Cartmel[2] wrote a paper claiming that for a maximum effect the arms of an interferometer should be inclined forty-five degrees to each other. The formula he derived gave the expected fringe shift as a function of the angle of drift. However, his formula seems to predict an effect even when the arms are taken parallel to each other. Accordingly, E. E. Haskins[3] made an investigation of the effect that may be expected when the arms are separated by an angle of forty-five degrees. He concluded that the expected maximum displacement of the fringes should be more than twice as great as the maximum displacement obtainable with an interferometer of similar size and arms at right angles to each other.

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1. Ether Drift Conference, Astrophys. J., <u>68</u>, 352, (1928)
2. Cartmel, W. B., Physical Review, <u>53</u>, 108, (1938)
3. Haskins, E. E., Doctorate Dissertation, Boston University, (1938)

PRELIMINARY REMARKS CONCERNING FIGURE I

Figure I is drawn entirely from the point of view of an observer fixed in the ether. Relative to this observer, the Michelson Interferometer is moving in a direction v, where the vector v is the horizontal component of the absolute velocity of the earth.

As the apparatus is set up by the moving observer the plate M is made to bisect the angle between the two arms S' and S" but due to the Lorentz contraction [1] the fixed observer will find that the two angles are not equal hence they have been designated by the different symbols θ and θ' and set equal after applying the Lorentz transformation.

For the general solution of this problem S' and S" have been made unequal. But, just as in the case of θ's the two arms may be set equal after the Lorentz transformations have been applied.

The mirror M' is adjusted by the moving observer to be perpendicular to the arm to which it is fixed but due to the Lorentz contraction of the mirror, the fixed observer will find that the angle between the normal to the mirror and the arm S' will be σ.

In order that the fringes observed in the telescope may be of a convenient width, the angle between the mirror M" and the arm to which it is fixed is made just slightly different from $\pi/2$. However, ω the angle which the fixed observer measures between the normal to the mirror and S" is

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1. The Lorentz transformation equations of the various angles and distances discussed in this section, will be derived on pages 18 to 30.

slightly modified because of Lorentz contraction of mirror M".

In the experiment, it is found that the telescope which is equally inclined to the two final rays must be focused on a point T which is in the vicinity of the mirror M'. Thus, the two paths of the light that must be compared are AC+CH-HT and AP+PL-LT.

Before proceeding with the main problem, it will be desirable to develop all the necessary equations of the Lorentz transformations, also the law of reflection from a moving mirror and the equation of the change of wave length upon reflection.

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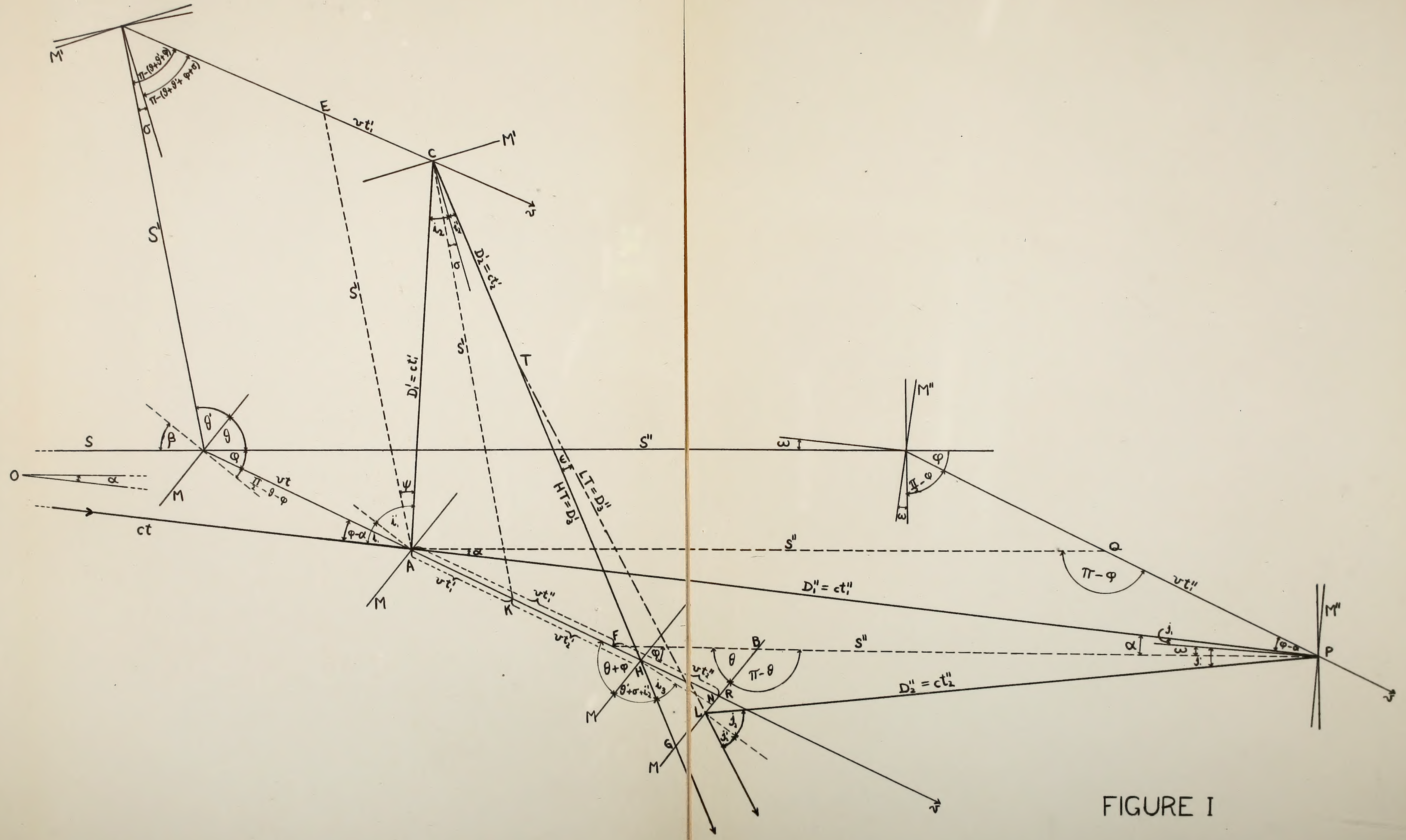

FIGURE I

LORENTZ TRANSFORMATION

Consider two inertial systems S and S_0, S being fixed relative to the ether and S_0 being fixed relative to the earth but moving relative to S. Throughout this work all angles and distances measured in the latter moving system will be designated by the subscript zero, whereas all angles and distances not bearing the subscript zero will refer to the former fixed system.

Let XYZ represent a rectangular system of coordinates fixed in the system S and $X_0Y_0Z_0$ a parallel set fixed in the system S_0. Let the two systems be oriented so that the velocity $\underline{v}$ of S_0 relative to S is in the positive X direction. Denote the coordinates and time as measured by an observer in S as x, y, z, t and as measured by observer in S_0 as x_0, y_0, z_0, t_0. Let the zero of time be determined by taking $t=t_0=0$ at the origins of the two systems when they are passing each other. Assuming these conditions, together with the hypothesis that the velocity of light has a constant value relative to all inertial systems, we obtain [1] the familiar Lorentz transformation equations:

$$x_0 = (x - vt)\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \quad ; \quad y = y_0 \quad \text{and} \quad z = z_0 \quad .$$

Let us now consider a rod such as a meter bar at rest in system S_0 with its axis parallel to the X_0 axis. Designate its two ends by the subscripts a and b.

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1 Richtmyer F. K. Introduction to Modern Physics p. 714 (1934) McGraw Hill Book Co.

Then:

$$x_{o_b} = (x_b - vt)(1-\frac{v^2}{c^2})^{-\frac{1}{2}}$$

and

$$x_{o_a} = (x_a - vt)(1-\frac{v^2}{c^2})^{-\frac{1}{2}}$$

provided that x_{o_b} and x_{o_a} are determined at the same time t_o.

Substracting, we obtain

$$x_b - x_a = (x_{o_b} - x_{o_a})(1-\frac{v^2}{c^2})^{\frac{1}{2}} \quad . \qquad \text{Eqn. (1)}$$

From this last equation we see that the measured length of the rod as determined in system S is less than that as determined in S_o according to the ratio

$$(1-\frac{v^2}{c^2})^{\frac{1}{2}} : 1 \quad .$$

From the equations involving y and z, it may be observed that there is no contraction along the y and z coordinates.

Thus, summing up, we may say that according to the Lorentz transformations there is a contraction along a line parallel to the direction of the velocity, but no contraction along a line perpendicular to the direction of the velocity.

Effect of Lorentz Transformation on Angles

Let us now consider the effect of the Lorentz transformation on angles. There are two cases that must be considered. First, angles of which one side is parallel to the direction of the velocity. The angles ϕ, $(\theta + \phi)$ and $(\theta + \theta' + \phi)$ are in this group. We shall refer to such angles as angles of Type I. The second type consists of angles of which neither side is parallel to the direction of the velocity. The angles θ and θ' are of this type. We shall refer to such angles as angles of Type II.

Angles of Type I

In fig. II, let EA and AC include an angle ϕ_0 as measured by observer moving in direction AC or $\underline{v}$. Due to the motion along $\underline{v}$, the component of AE parallel to $\underline{v}$ will undergo a contraction and the angle that will be measured by the observer fixed in the ether will be DAB or ϕ.

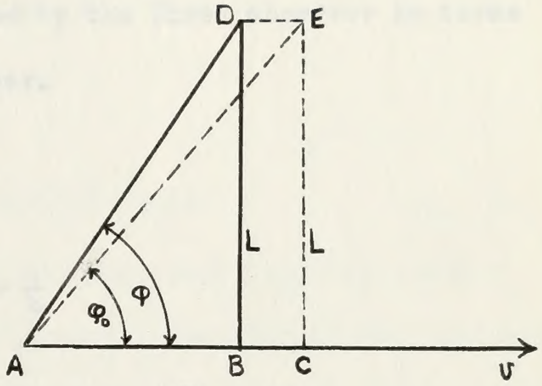

Fig. II

Let L be component of AD perpendicular to $\underline{v}$

Thus:

$$AC = L \operatorname{ctn} \varphi_0$$

and from equation (1)

$$AB = L \operatorname{ctn} \varphi_0 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} .$$

Also

$$\tan \varphi = L/AB$$

$$= L / L \operatorname{ctn} \varphi_0 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$$

$$= \tan \varphi_0 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} .$$

Expand and drop all terms involving powers of v/c greater than the second. Throughout this work, we shall not keep higher powers than the second.

Therefore:

$$\tan \varphi = \tan \varphi_0 \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right) . \qquad \text{Eqn. (2)}$$

In the following work we shall deal almost exclusively with sines and cosines. Therefore it will be convenient to derive equations of the sines and cosines of the angles as measured by the fixed observer in terms of angles as measured by the moving observer.

$$\cos\varphi = (1+\tan^2\varphi)^{-\frac{1}{2}}$$

$$= \left[1+\tan^2\varphi_0\left(1+\frac{v^2}{c^2}\right)\right]^{-\frac{1}{2}}$$

$$= \frac{1}{\left(\sec^2\varphi_0 + \frac{v^2}{c^2}\tan^2\varphi_0\right)^{\frac{1}{2}}}$$

Multiply numerator and denominator by $\cos\phi_0$.

$$\cos\varphi = \frac{\cos\varphi_0}{\left(1+\frac{v^2}{c^2}\sin^2\varphi_0\right)^{\frac{1}{2}}}$$

Expand in terms of powers of v/c.

$$\cos\varphi = \cos\varphi_0\left(1-\frac{1}{2}\frac{v^2}{c^2}\sin^2\varphi_0+\cdots\right) \qquad \text{Eqn. (3)}$$

Also

$$\sin\varphi = \cos\varphi\tan\varphi$$

$$\sin\varphi = \sin\varphi_0\left(1+\frac{1}{2}\frac{v^2}{c^2}\cos^2\varphi_0+\cdots\right). \qquad \text{Eqn. (4)}$$

Similarly replacing ϕ by $(\theta + \phi)$ and ϕ_0 by $(\theta_0 + \phi_0)$ we obtain

$$\sin(\vartheta+\varphi) = \sin(\vartheta_0+\varphi_0)\left[1+\frac{1}{2}\frac{v^2}{c^2}\cos^2(\vartheta_0+\varphi_0)+\cdots\right]. \qquad \text{Eqn. (5)}$$

and

$$\cos(\vartheta+\varphi) = \cos(\vartheta_0+\varphi_0)\left[1-\frac{1}{2}\frac{v^2}{c^2}\sin^2(\vartheta_0+\varphi_0)+\cdots\right]. \qquad \text{Eqn. (6)}$$

Also, replacing ϕ by $(\theta + \theta' + \phi)$ and ϕ_0 by $(2\theta_0 + \phi_0)$ we get

$$\sin(\vartheta+\vartheta'+\varphi) = \sin(2\vartheta_0+\varphi_0)\left[1+\frac{1}{2}\frac{v^2}{c^2}\cos^2(2\vartheta_0+\varphi_0)+\cdots\right] \qquad \text{Eqn. (7)}$$

and

$$\cos(\vartheta+\vartheta'+\varphi) = \cos(2\vartheta_0+\varphi_0)\left[1-\frac{1}{2}\frac{v^2}{c^2}\sin^2(2\vartheta_0+\varphi_0)+\cdots\right]. \qquad \text{Eqn. (8)}$$

It will be observed that if any of these angles are set equal to 180 degrees, its value undergoes no change upon transformation. However, this statement is not true of angles of 90 degrees.

Angles of Type II

To determine the sine and the cosine of angles of Type II in terms of angles measured by an observer in the moving system, let us consider the identity:

$$\vartheta = (\vartheta+\varphi) - \varphi$$

then:

$$\sin\vartheta = \sin[(\vartheta+\varphi)-\varphi]$$

$$= \sin(\vartheta+\varphi)\cos\varphi - \cos(\vartheta+\varphi)\sin\varphi$$

Substitute equations 3, 4, 5, and 6.

$$\sin\vartheta = \sin(\vartheta_0+\varphi_0)\cos\varphi_0\left[1+\frac{1}{2}\frac{v^2}{c^2}\cos^2(\vartheta_0+\varphi_0)\right]\left[1-\frac{1}{2}\frac{v^2}{c^2}\sin^2\varphi_0\right]$$

$$- \cos(\vartheta_0+\varphi_0)\sin\varphi_0\left[1-\frac{1}{2}\frac{v^2}{c^2}\sin^2(\vartheta_0+\varphi_0)\right]\left[1+\frac{1}{2}\frac{v^2}{c^2}\cos^2\varphi_0\right].$$

Multiply and arrange according to powers of v/c.

$$\sin\vartheta = \sin(\vartheta_0+\varphi_0)\cos\varphi_0 - \cos(\vartheta_0+\varphi_0)\sin\varphi_0$$

$$+\frac{1}{2}\frac{v^2}{c^2}[\sin(\vartheta_0+\varphi_0)\cos\varphi_0\cos^2(\vartheta_0+\varphi_0) - \sin(\vartheta_0+\varphi_0)\cos\varphi_0\sin^2\varphi_0$$

$$+\cos(\vartheta_0+\varphi_0)\sin^2(\vartheta_0+\varphi_0)\sin\varphi_0 - \cos(\vartheta_0+\varphi_0)\sin\varphi_0\cos^2\varphi_0]$$

Replace $\cos^2(\theta_0 + \phi_0)$ by $1 - \sin^2(\theta_0 + \phi_0)$ and $\sin^2\phi_0$ by $(1 - \cos^2\phi_0)$.

Then

$$\sin\vartheta = \sin\vartheta_0 + \frac{1}{2}\frac{v^2}{c^2}[\sin(\vartheta_0+\varphi_0)\cos\varphi_0 - \sin^3(\vartheta_0+\varphi_0)\cos\varphi_0 - \sin(\vartheta_0+\varphi_0)\cos\varphi_0 + \sin(\vartheta_0+\varphi_0)\cos^3\varphi_0 + \cos(\vartheta_0+\varphi_0)\sin^2(\vartheta_0+\varphi_0)\sin\varphi_0 - \cos(\vartheta_0+\varphi_0)\sin\varphi_0\cos^2\varphi_0]$$

$$= \sin\vartheta_0 + \frac{1}{2}\frac{v^2}{c^2}[-\sin^2(\vartheta_0+\varphi_0)\sin\vartheta_0 + \sin\vartheta_0\cos^2\varphi_0]$$

$$\sin\vartheta = \sin\vartheta_0[1 - \frac{1}{2}\frac{v^2}{c^2}\{\sin^2(\vartheta_0+\varphi_0) - \cos^2\varphi_0\}]$$ Eqn. (9)

Again, let us assume the identity:

$$\vartheta = (\vartheta+\varphi) - \varphi$$

then:

$$\cos\vartheta = \cos[(\vartheta+\varphi) - \varphi]$$
$$= \cos(\vartheta+\varphi)\cos\varphi + \sin(\vartheta+\varphi)\sin\varphi$$

Substitute equations 3, 4, 5, and 6.

$$\cos\vartheta = \cos(\vartheta_0+\varphi_0)\cos\varphi_0[1 - \frac{1}{2}\frac{v^2}{c^2}\sin^2(\vartheta_0+\varphi_0)][1 - \frac{1}{2}\frac{v^2}{c^2}\sin^2\varphi_0] + \sin(\vartheta_0+\varphi_0)\sin\varphi_0[1 + \frac{1}{2}\frac{v^2}{c^2}\cos^2(\vartheta_0+\varphi_0)][1 + \frac{1}{2}\frac{v^2}{c^2}\cos^2\varphi_0]$$

Multiply and arrange according to powers of v/c.

$$\cos\vartheta = \cos(\vartheta_0+\varphi_0)\cos\varphi_0 + \sin(\vartheta_0+\varphi_0)\sin\varphi_0 + \frac{1}{2}\frac{v^2}{c^2}[-\cos(\vartheta_0+\varphi_0)\cos\varphi_0\sin^2\varphi_0 - \cos(\vartheta_0+\varphi_0)\sin^2(\vartheta_0+\varphi_0)\cos\varphi_0 + \sin(\vartheta_0+\varphi_0)\sin\varphi_0\cos^2\varphi_0 + \sin(\vartheta_0+\varphi_0)\cos^2(\vartheta_0+\varphi_0)\sin\varphi_0]$$

Replace $\sin^2(\theta_0 + \phi_0)$ by $1 - \cos^2(\theta_0 + \phi_0)$ and $\sin^2\phi_0$ by $1 + \cos^2\phi_0$.

Then $$\cos\vartheta = \cos\vartheta_0 + \frac{1}{2}\frac{v^2}{c^2}\Big[-\cos(\vartheta_0+\varphi_0)\cos\varphi_0 + \cos(\vartheta_0+\varphi_0)\cos^3\varphi_0 - \cos(\vartheta_0+\varphi_0)\cos\varphi_0 + \cos^3(\vartheta_0+\varphi_0)\cos\varphi_0 + \sin(\vartheta_0+\varphi_0)\sin\varphi_0\cos^2\varphi_0 + \sin(\vartheta_0+\varphi_0)\cos^2(\vartheta_0+\varphi_0)\sin\varphi_0\Big]$$

$$= \cos\vartheta_0 + \frac{1}{2}\frac{v^2}{c^2}\Big[\cos^2(\vartheta_0+\varphi_0)\cos\vartheta_0 + \cos^2\varphi_0\cos\vartheta_0 - 2\cos(\vartheta_0+\varphi_0)\cos\varphi_0\Big]$$

$$= \cos\vartheta_0 + \frac{1}{2}\frac{v^2}{c^2}\Big[\cos^2(\vartheta_0+\varphi_0)\cos\vartheta_0 + \cos\varphi_0\{\cos\varphi_0\cos\vartheta_0 - 2\cos\vartheta_0\cos\varphi_0 + 2\sin\vartheta_0\sin\varphi_0\}\Big]$$

$$\cos\vartheta = \cos\vartheta_0\Big[1 + \frac{1}{2}\frac{v^2}{c^2}\{\cos^2(\vartheta_0+\varphi_0) + \cos\varphi_0(2\tan\vartheta_0\sin\varphi_0 - \cos\varphi_0)\} + \cdots\Big]$$

Eqn. (10)

Similarly, to obtain equations for θ' replace ϕ_0 by $(\theta_0 + \phi_0)$.

This yields

$$\sin\vartheta' = \sin\vartheta_0\Big[1 - \frac{1}{2}\frac{v^2}{c^2}\{\sin^2(2\vartheta_0+\varphi_0) - \cos^2(\vartheta_0+\varphi_0)\} + \cdots\Big]$$

Eqn. (11)

and

$$\cos\vartheta' = \cos\vartheta_0\Big[1 + \frac{1}{2}\frac{v^2}{c^2}\{\cos^2(2\vartheta_0+\varphi_0) + \cos(\vartheta_0+\varphi_0)[2\tan\vartheta_0\sin(\vartheta_0+\varphi_0) - \cos(\vartheta_0+\varphi_0)]\} + \cdots\Big]$$

Eqn. (12)

In later work, we shall find that the angle (θ - θ') occurs frequently, therefore it well be convenient to derive the value of sin(θ-θ') and cos(θ-θ') at this point.

$$\sin(\vartheta-\vartheta') = \sin\vartheta\cos\vartheta' - \cos\vartheta\sin\vartheta'$$

Substitute equations 9, 10, 11, and 12.

$$\sin(\vartheta-\vartheta') = \sin\vartheta_0\cos\vartheta_0\Big[1 - \frac{1}{2}\frac{v^2}{c^2}\{\sin^2(\vartheta_0+\varphi_0) - \cos^2\varphi_0 - \cos^2(2\vartheta_0+\varphi_0) - 2\tan\vartheta_0\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0) + \cos^2(\vartheta_0+\varphi_0)\}\Big]$$
$$- \sin\vartheta_0\cos\vartheta_0\Big[1 + \frac{1}{2}\frac{v^2}{c^2}\{\cos^2(\vartheta_0+\varphi_0) + 2\tan\vartheta_0\sin\varphi_0\cos\varphi_0 - \cos^2\varphi_0 - \sin^2(2\vartheta_0+\varphi_0) + \cos^2(\vartheta_0+\varphi_0)\}\Big]$$

Multiply and arrange according to powers of v/c.

$$\sin(\vartheta-\vartheta') = \frac{1}{2}\frac{v^2}{c^2}\sin\vartheta_0\cos\vartheta_0\{-\sin^2(\vartheta_0+\varphi_0)+2\cos^2\varphi_0+\cos^2(2\vartheta_0+\varphi_0)-2\cos^2(\vartheta_0+\varphi_0)+\sin^2(2\vartheta_0+\varphi_0)$$
$$-\cos^2(\vartheta_0+\varphi_0)+2\tan\vartheta_0[\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)-\sin\varphi_0\cos\varphi_0]\}$$

$$\sin(\vartheta-\vartheta') = \frac{v^2}{c^2}\sin\vartheta_0\cos\vartheta_0\{\cos^2\varphi_0-\cos^2(\vartheta_0+\varphi_0)+\tan\vartheta_0[\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)-\sin\varphi_0\cos\varphi_0]\}+\cdots \qquad \text{Eqn. (13)}$$

Also

$$\cos(\vartheta-\vartheta') = [1-\sin^2(\vartheta-\vartheta')]^{\frac{1}{2}}$$

$$\cos(\vartheta-\vartheta') = 1+\cdots \qquad \text{Eqn. (14)}$$

neglecting terms involving powers of v/c greater than the second.

It will be desirable now to determine the value of the sine and the cosine of angle σ. The nature of the angle σ was discussed on page 15.

Let fig. IIIa represent M' as seen by the observer fixed in the ether and fig. IIIb, M' as seen by the moving observer. It is clear that

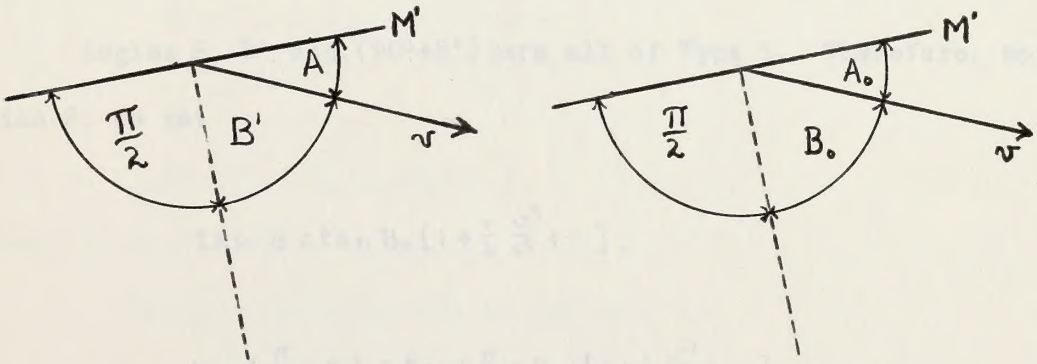

Fig. IIIa   Fig. IIIb

due to the contraction of the mirror in the direction of v the mirror suffers a slight rotation and therefore the normals constructed to the mirror by the two observers will be different. The angle between these two normals is called σ.

In fig. IIIa the dotted line represents the normal to the mirror

constructed by the fixed observer. In his coordinate system $A+B'=90^{\circ}$. Similarly in fig. IIIb the dotted line represents the normal to the mirror constructed by the moving observer. In his coordinate system $A_0+B_0=90^{\circ}$. Referring to figure I, we can deduce that

$$B_0 = (\pi - 2\vartheta_0 - \varphi_0).$$

The angle B_0 becomes angle B' in the fixed observer's coordinate system.

Thus
$$B' = (\pi - \vartheta - \vartheta' - \varphi).$$

However, in figure I we have designated this same angle as

$$B = (\pi - \vartheta - \vartheta' - \varphi - \sigma).$$

Therefore, we have

$$B' = B + \sigma.$$

Angles B, B' and $(90^{\circ}+B')$ are all of Type I. Therefore, applying equation 2, we get

$$\tan B = \tan B_0 [1 + \tfrac{1}{2}\tfrac{v^2}{c^2} + \cdots].$$

Also

$$\tan(\tfrac{\pi}{2} + B') = \tan(\tfrac{\pi}{2} + B_0)[1 + \tfrac{1}{2}\tfrac{v^2}{c^2} + \cdots]$$

or

$$\operatorname{ctn} B' = \operatorname{ctn} B_0 [1 + \tfrac{1}{2}\tfrac{v^2}{c^2} + \cdots].$$

Since

$$B' = B + \sigma$$

$$\operatorname{ctn} B' = \operatorname{ctn}(B+\sigma) = \operatorname{ctn} B_0 [1 + \tfrac{1}{2}\tfrac{v^2}{c^2} + \cdots].$$

Let us expand $\operatorname{ctn}(B+\sigma)$.

$$\frac{\operatorname{ctn} B \operatorname{ctn} \sigma - 1}{\operatorname{ctn} \sigma + \operatorname{ctn} B} = \operatorname{ctn} B_0 [1 + \tfrac{1}{2} \tfrac{v^2}{c^2} + \cdots]$$

Now,

$$\operatorname{ctn} B = \frac{1}{\tan B} = \operatorname{ctn} B_0 [1 - \tfrac{1}{2} \tfrac{v^2}{c^2} + \cdots]$$

Substitute the latter expression in the equation preceeding it.

$$\frac{\operatorname{ctn} B_0 \operatorname{ctn} \sigma [1 - \frac{1}{2} \frac{v^2}{c^2}] - 1}{\operatorname{ctn} \sigma + \operatorname{ctn} B_0 [1 - \frac{1}{2} \frac{v^2}{c^2}]} = \operatorname{ctn} B_0 [1 + \tfrac{1}{2} \tfrac{v^2}{c^2}]$$

Cross multiply:

$$\operatorname{ctn} B_0 \operatorname{ctn} \sigma - \frac{1}{2} \frac{v^2}{c^2} \operatorname{ctn} B_0 \operatorname{ctn} \sigma - 1 = \operatorname{ctn} B_0 \operatorname{ctn} \sigma + \frac{1}{2} \frac{v^2}{c^2} \operatorname{ctn} B_0 \operatorname{ctn} \sigma + \operatorname{ctn}^2 B_0$$

From which we obtain

$$\operatorname{ctn} \sigma = - \frac{1}{\frac{v^2}{c^2} \sin^2 B_0 \operatorname{ctn} B_0}$$

and

$$\tan \sigma = - \frac{v^2}{c^2} \sin B_0 \cos B_0$$

Substitue for B_0 its value $(\pi - 2\theta_0 - \phi_0)$ then,

$$\tan \sigma = - \frac{v^2}{c^2} \sin(\pi - 2\vartheta_0 - \varphi_0) \cos(\pi - 2\vartheta_0 - \varphi_0)$$

$$\tan \sigma = \frac{v^2}{c^2} \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0).$$

Also

$$\cos \sigma = (1 + \tan^2 \sigma)^{-\frac{1}{2}} = 1 + \cdots \qquad \text{Eqn. (15)}$$

neglecting powers of v/c greater than the second.

and

$$\sin \sigma = \cos \sigma \tan \sigma = \frac{v^2}{c^2} \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \qquad \text{Eqn. (16)}$$

We shall now determine the sine and cosine of angle ω. The nature of this angle was discussed on page 15 .

Let figure IVa represent M'' as seen by the observer fixed in the ether and figure IVb, M" as seen by the moving observer. As in the case of

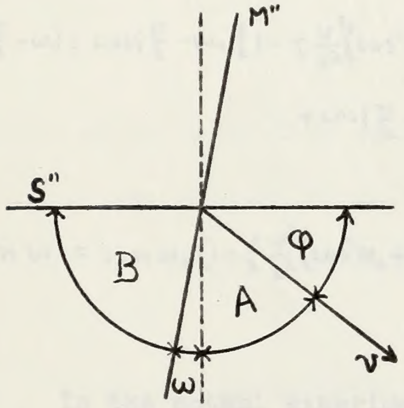

Fig. IVa

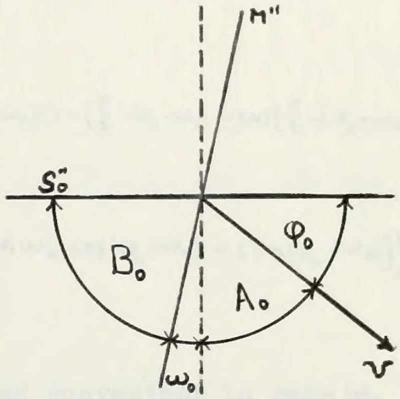

Fig. IVb

the mirror M', the mirror M" undergoes a contraction in the direction of v. The effect of this contraction is that the mirror is slightly rotated.

As was explained on page 15, a small angle must be introduced at M" so that the fringes observed in the telescope may be adjusted to some convenient width. Let ω represent the angle introduced in the adjustment of the fringes as measured by the observer fixed in the ether, plus the small angle which comes into existence from the change of orientation of the mirror caused by the contraction of M".

In figure IVa , the dotted line represents the normal to S" constructed by the observer fixed in the ether. In his coordinate system $\omega = 90^\circ - B$.

In figure IVb, the dotted line represents the normal to S''_0 constructed by the moving observer. In his coordinate system, $\omega_0 = 90^\circ - B_0$.

Also $A_0 = 90^\circ - \varphi_0 + \omega_0$ and $B_0 = 90^\circ - \omega_0$.

By equation 10

$$\cos B = \cos B_0\left[1 - \frac{1}{2}\frac{v^2}{c^2}\{\cos^2(A_0+B_0) + \cos A_0[2\tan B_0 \sin A_0 - \cos A_0]\} + \cdots\right]$$

Substitute values of B, B_0 and A_0.

$$\cos(\tfrac{\pi}{2}-\omega) = \cos(\tfrac{\pi}{2}-\omega_0)\Big[1 - \frac{1}{2}\frac{v^2}{c^2}\Big(\cos^2(\pi-\varphi_0)$$

$$+ \cos\{\tfrac{\pi}{2}-(\varphi_0-\omega_0)\}[2\tan(\tfrac{\pi}{2}-\omega_0)\sin\{\tfrac{\pi}{2}-(\varphi_0-\omega_0) - \cos\{\tfrac{\pi}{2}-(\varphi_0-\omega_0)\}]\Big) + \cdots\Big]$$

Or

$$\sin\omega = \sin\omega_0\Big[1 - \frac{1}{2}\frac{v^2}{c^2}\Big(\cos^2\varphi_0 + \sin(\varphi_0-\omega_0)\{2\operatorname{ctn}\omega_0\cos(\varphi_0-\omega_0) - \sin(\varphi_0-\omega_0)\}\Big) + \cdots\Big].$$

In the actual experiment it is found convenient to make ω_0 of the order of magnitude of v/c. Therefore, we may set

$$\sin\omega_0 = \frac{v}{c}\sin\mu_0$$

where μ_0 is defined in terms of this equation.

$$\cos\omega_0 = (1-\sin^2\omega_0)^{\frac{1}{2}}$$

$$= 1 - \frac{1}{2}\frac{v^2}{c^2}\sin^2\mu_0 + \cdots$$

In equation for $\sin\omega$ we observe that $\sin(\phi_0-\omega_0)$, $\cos(\phi_0-\omega_0)$ and $\operatorname{ctn}\omega_0$ occur in the coefficient of $(v/c)^2$. Therefore, it will be necessary to expand them only up to $(v/c)^{-1}$ after substituting for $\sin\omega_0$. Thus, all that remains in the order term is

$$\operatorname{ctn}\omega_0 = \frac{\cos\omega_0}{\sin\omega_0} = \frac{1}{\frac{v}{c}\sin\mu_0} + \cdots$$

Substituting this value, we get

$$\sin\omega = \frac{v}{c}\sin\mu_0\left[1 - \frac{1}{2}\frac{v^2}{c^2}\sin\varphi_0\left\{\frac{2\cos\varphi_0}{\frac{v}{c}\sin\mu_0}\right\} + \cdots\right]$$

Or:

$$\sin\omega = \frac{v}{c}\sin\mu_0 + \frac{v^2}{c^2}\sin\varphi_0\cos\varphi_0 + \cdots \qquad \text{Eqn. (17)}$$

And;

$$\cos\omega = (1-\sin^2\omega)^{\frac{1}{2}}$$
$$= 1 - \frac{1}{2}\frac{v^2}{c^2}\sin^2\mu_0 + \cdots \qquad \text{Eqn. (18)}$$

It is clear from equation 17 that the v/c term is the part of the angle whcih was introduced in the adjustment of the mirror; whereas the $(v/c)^2$ term is the part introduced because of the contraction of the mirror.

Effect of Lorentz Transformation on Distances

Referring to fig. I we observe that the angle between S' and the direction of the velocity is $(\theta+\theta'+\phi)$. Therefore the component of the velocity parallel to S' is v $\cos(\theta+\theta'+\phi)$. Substituting this value of v in equation 1 gives:

$$S' = S_0'\left[1-\frac{v^2}{c^2}\cos^2(\vartheta+\vartheta'+\varphi)+\cdots\right]^{\frac{1}{2}}$$

Substitute equation 8 and expand.

$$S' = S_0'\left[1-\frac{1}{2}\frac{v^2}{c^2}\cos^2(2\vartheta_0+\varphi_0)+\cdots\right] \qquad \text{Eqn. (19)}$$

Similarly, referring to fig. I we observe that the angle between S" and the direction of the velocity is ϕ. Therefore, the component of the velocity parallel to S" is v $\cos\phi$. Substituting this value of v in equation 1 gives;

$$S'' = S_0''\left[1-\frac{v^2}{c^2}\cos^2\varphi\right]^{\frac{1}{2}}.$$

Substitute equation 3 and expand.

$$S'' = S_0''\left[1-\frac{1}{2}\frac{v^2}{c^2}\cos^2\varphi_0+\cdots\right] \qquad \text{Eqn. (20)}$$

LAW OF REFLECTION FROM A MOVING MIRROR

Consider a mirror M moving in a given direction with a velocity v, the angle between the direction v and the mirror being ϕ. Let AB be an

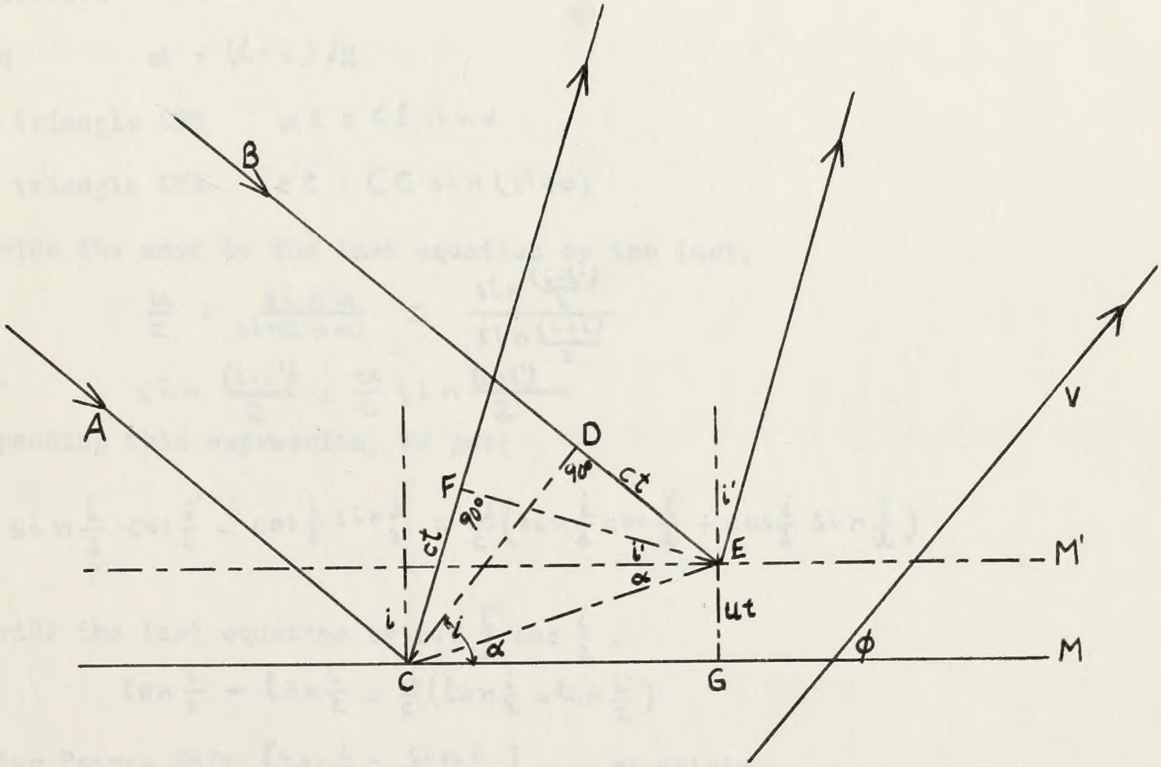

Fig. V

incident beam of parallel light of which CD represents the wave front at the time that the ray AC strikes the mirror M. Let i be the angle of incidence, that is, the angle between the normal to the mirror and the incident ray of light. While the ray of light BD advances a distance ct to E, the mirror has advanced a distance ut, where u=v sin ϕ and c stands for the velocity of light. This position is represented by M'. During the interval of time that the ray BD traverses the distance DE, the ray reflected at C

traverses an equal distance CF=ct. Construct EF and CF so that they include a right angle. EF represents the wave front of the reflected ray. Let i' be the angle of reflection, and α the angle between CE and the mirror.

The right triangles CFE and CDE are conguent since they have a common hypotenuse and equal legs CF and ED.

Therefore $\angle i-\alpha = \angle i'+\alpha$

and $\alpha = (i-i')/2$

In triangle CEG $ut = CE\sin\alpha$

In triangle CFE $ct = CE\sin(i'+\alpha)$

Divide the next to the last equation by the last.

$$\frac{u}{c} = \frac{\sin\alpha}{\sin(i'+\alpha)} = \frac{\sin\frac{(i-i')}{2}}{\sin\frac{(i+i')}{2}}$$

or,

$$\sin\frac{(i-i')}{2} = \frac{u}{c}\sin\frac{(i+i')}{2}$$

Expanding this expression, we get:

$$\sin\frac{i}{2}\cos\frac{i'}{2} - \cos\frac{i}{2}\sin\frac{i'}{2} = \frac{u}{c}\left(\sin\frac{i}{2}\cos\frac{i'}{2} + \cos\frac{i}{2}\sin\frac{i'}{2}\right)$$

Divide the last equation by $\cos\frac{i'}{2}\cos\frac{i}{2}$.

$$\tan\frac{i}{2} - \tan\frac{i'}{2} = \frac{u}{c}\left(\tan\frac{i}{2} + \tan\frac{i'}{2}\right)$$

Using Peirce #578 $\left[\tan\frac{i}{2} = \frac{\sin i}{1+\cos i}\right]$, we obtain

$$\frac{\sin i}{1+\cos i} - \frac{\sin i'}{1+\cos i'} = \frac{u}{c}\frac{\sin i}{1+\cos i} + \frac{u}{c}\frac{\sin i'}{1+\cos i'}$$

Collect terms and replace cos i' by $(1-\sin^2 i')^{\frac{1}{2}}$.

$$\frac{\sin i'}{1+(1-\sin^2 i')^{1/2}} = A\frac{\sin i}{1-\cos i}$$

Where A is defined as (c-u)/(c+u).

Cross multiply and rearrange the last expression.

$$\sin i' + \sin i\cos i' - A\sin i = A\sin i(1-\sin^2 i')^{\frac{1}{2}}$$

Squaring both sides yields:

$$\sin i' + \sin i' \cos^2 i + 2 \sin i' \cos i - 2A \sin i - 2 \sin i \cos i + A^2 \sin^2 i = 0$$

From which we obtain

$$\sin i' = \frac{2A \sin i}{1 + \cos i + A^2(1 - \cos i)}$$

Substitute for A, it's value and multiply through by (c + u).

$$\sin i' = \frac{2(c^2 - u^2)\sin i}{(c+u)^2(1+\cos i) + (c-u)^2(1-\cos i)}$$

Expand and divide through by c^2.

$$\sin i' = \frac{\left(1 - \frac{u^2}{c^2}\right)\sin i}{1 + 2\frac{u}{c}\cos i + \frac{u^2}{c^2}}$$

Divide numerator by denominator and neglect powers of u/c greater than the second. This yields:

$$\sin i' = \sin i - 2\frac{u}{c}\sin i \cos i + 2\frac{u^2}{c^2}\sin i (2\cos^2 i - 1) + \cdots$$

Using the relationship $\cos i' = (1 - \sin^2 i')^{\frac{1}{2}}$, we obtain

$$\cos i' = \cos i + 2\frac{u}{c}\sin^2 i - 4\frac{u^2}{c^2}\sin^2 i \cos i + \cdots$$

Replacing u by its equal $v \sin \phi$, we obtain as the final equations for sin i' and cos i' :

$$\sin i' = \sin i - 2\frac{v}{c}\sin\varphi \sin i \cos i + 2\frac{v^2}{c^2}\sin^2\varphi \sin i (2\cos^2 i - 1) \cdots \quad \text{Eqn. (21)}$$

$$\cos i' = \cos i + 2\frac{v}{c}\sin\varphi \sin^2 i - 4\frac{v^2}{c^2}\sin^2\varphi \sin^2 i \cos i + \cdots \quad \text{Eqn. (22)}$$

We have thus expressed both the sine and the cosine of the angle of reflection in terms of the sine and the cosine of the angle of incidence;

the sine of the angle between the mirror and the direction of the velocity; the velocity of light and the velocity of the mirror, this latter velocity being positive when the mirror is approaching the source of light and negative when receding from the source of light.

CHANGE OF WAVE LENGTH UPON REFLECTION FROM A MOVING MIRROR

In figure VI, CO represents a ray of light incident upon the mirror M at O. The lines drawn perpendicular to this ray represent succesive wave fronts one wave length apart. Similarly, the lines drawn perpendicular to the reflected ray OD represent reflected wave fronts one wave length apart.

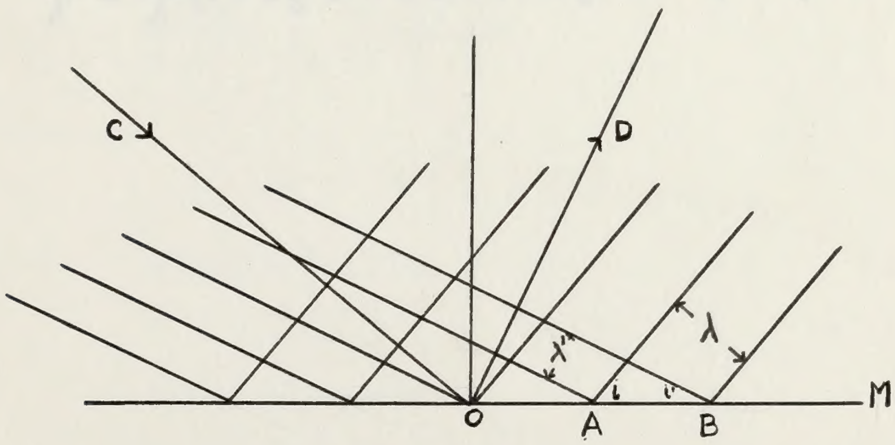

Fig. VI

If i is the angle of incidence, i' the angle of reflection, λ the wave length of the incident light and λ' the wave length of the reflected light, we have:

$$\lambda' = AB \sin i'$$

and

$$\lambda = AB \sin i .$$

Dividing the former equation by the latter, gives:

$$\frac{\lambda'}{\lambda} = \frac{\sin i'}{\sin i}$$

Substituting equation 21 and dividing by sin i, we obtain:

$$\lambda' = \lambda\left[1 - 2\frac{v}{c}\sin\varphi\cos i + 2\frac{v^2}{c^2}\sin^2\varphi\,(2\cos^2 i - 1) + \cdots\right].$$

Since we shall have to divide the various distances by the corresponding wave lengths, it will be desirable to obtain the reciprocal of the above equation. Thus,

$$\frac{1}{\lambda'} = \frac{1}{\lambda}\left[1 + 2\frac{v}{c}\sin\varphi\cos i + 2\frac{v^2}{c^2}\sin^2\varphi + \cdots\right] \qquad \text{Eqn. (23)}$$

ANGLE OF REFLECTION OF THE RAY FROM PLATE M ALONG ARM S'

In figure I the apparatus is represented as moving in a general horizontal direction $\underline{v}$. Because of this motion, the ray of light leaving the source at O is not incident upon the plate M in a direction parallel to the source arm S.

If α is the angle between the incident ray of light and S, we have by the sine law:

$$\frac{\sin\alpha}{vt} = \frac{\sin(\pi-\varphi)}{ct}.$$

from which we get:

$$\sin\alpha = \frac{v}{c}\sin\varphi \qquad \text{Eqn. (24)}$$

and

$$\cos\alpha = (1-\sin^2\alpha)^{\frac{1}{2}}$$

$$\cos\alpha = 1-\frac{1}{2}\frac{v^2}{c^2}\sin^2\varphi+\cdots, \qquad \text{Eqn. (25)}$$

If β is the angle between the normal to M and the source arm S, and i_1 angle of incidence at the plate M then:

$$i_1 = \beta-\alpha.$$

Also $$\beta = \frac{\pi}{2}-\theta.$$

Therefore, substituting for β, we get:

$$i_1 = \frac{\pi}{2}-\theta-\alpha \qquad \text{Eqn. (26)}$$

To find the sine and the cosine of the angle of reflection, apply equations 21 and 22. Since the mirror is receding from the source of light the velocity is negative. The angle between the plate M and the direction of the velocity is $(\theta+\phi)$. Thus $v \sin \phi$ must be replaced by $-v \sin (\theta+\phi)$. Hence, substituting in equations 21 and 22, we get:

$$\sin i_1' = \sin i_1 + 2\frac{v}{c}\sin(\vartheta+\varphi)\sin i_1 \cos i_1 + 2\frac{v^2}{c^2}\sin^2(\vartheta+\varphi)\sin i_1(2\cos^2 i_1 - 1)+\cdots, \qquad \text{Eqn. (27)}$$

and

$$\cos i_1' = \cos i_1 - 2\frac{v}{c}\sin(\vartheta+\varphi)\sin^2 i_1 - 4\frac{v^2}{c^2}\sin^2(\vartheta+\varphi)\sin^2 i_1 \cos i_1 + \cdots . \qquad \text{Eqn. (28)}$$

Substituting equation 26 we get:

$$\sin i_1 = \cos(\vartheta+\alpha) = \cos\vartheta\cos\alpha - \sin\vartheta\sin\alpha .$$

Substitute equations 24 and 25 .

$$\sin i_1 = \cos\vartheta - \frac{v}{c}\sin\vartheta\sin\varphi - \frac{1}{2}\frac{v^2}{c^2}\cos\vartheta\sin^2\varphi + \cdots .$$

Again, substituting equation 26, we get:

$$\cos i_1 = \sin(\vartheta+\alpha) = \sin\vartheta\cos\alpha + \cos\vartheta\sin\alpha .$$

Substitute equations 24, 25 .

$$\cos i_1 = \sin\vartheta + \frac{v}{c}\cos\vartheta\sin\varphi - \frac{1}{2}\frac{v^2}{c^2}\sin\vartheta\sin^2\varphi + \cdots .$$

The following expressions will be developed only up to the power of $\frac{v}{c}$ to which they will be needed when they are substituted in equations 27 and 28.

$$\sin i_1 \cos i_1 = \sin\vartheta\cos\vartheta + \frac{v}{c}\sin\varphi(2\cos^2\vartheta - 1) + \cdots$$

$$\sin^2 i_1 = \cos^2\vartheta - 2\frac{v}{c}\sin\varphi\sin\vartheta\cos\vartheta + \cdots$$

$$\sin i_1 \cos^2 i_1 = \sin^2\vartheta\cos\vartheta + \cdots$$

$$\sin^2 i_1 \cos i_1 = \sin\vartheta\cos^2\vartheta + \cdots$$

Substituting the appropriate expressions in equation 27, we obtain:

$$\begin{aligned}\sin i_1' = {} & \cos\vartheta - \frac{v}{c}\sin\varphi\sin\vartheta - \frac{1}{2}\frac{v^2}{c^2}\sin^2\varphi\cos\vartheta \\ & + 2\frac{v}{c}\sin(\vartheta+\varphi)[\sin\vartheta\cos\vartheta + \frac{v}{c}\sin\varphi(2\cos^2\vartheta - 1)] \\ & + 2\frac{v^2}{c^2}\sin^2(\vartheta+\varphi)[2\sin^2\vartheta\cos\vartheta - \cos\vartheta] + \cdots\end{aligned}$$

Arranging according to powers of v/c, we get:

$$\begin{aligned}\sin i_1' = {} & \cos\vartheta + \frac{v}{c}\sin\vartheta[2\sin(\vartheta+\varphi)\cos\vartheta - \sin\varphi] \\ & + \frac{v^2}{c^2}[2\sin(\vartheta+\varphi)\sin\varphi(2\cos^2\vartheta - 1) - \frac{1}{2}\sin^2\varphi\cos\vartheta \\ & + 2\sin^2(\vartheta+\varphi)\cos\vartheta(2\sin^2\vartheta - 1)] + \cdots\end{aligned} \qquad \text{Eqn. (29)}$$

Also, substituting the appropriate expressions in equation 28, we obtain:

$$\begin{aligned}\cos i_1' = {} & \sin\vartheta + \frac{v}{c}\sin\varphi\cos\vartheta - \frac{1}{2}\frac{v^2}{c^2}\sin^2\varphi\sin\vartheta \\ & - 2\frac{v}{c}\sin(\vartheta+\varphi)[\cos^2\vartheta - 2\frac{v}{c}\sin\varphi\sin\vartheta\cos\vartheta] \\ & - 4\frac{v^2}{c^2}\sin^2(\vartheta+\varphi)\sin\vartheta\cos^2\vartheta + \cdots\end{aligned}$$

Arranging according to powers of v/c, we get:

$$\begin{aligned}\cos i_1' = {} & \sin\vartheta + \frac{v}{c}\cos\vartheta[\sin\varphi - 2\cos\vartheta\sin(\vartheta+\varphi)] \\ & + \frac{v^2}{c^2}[4\sin(\vartheta+\varphi)\sin\varphi\sin\vartheta\cos\vartheta - \frac{1}{2}\sin^2\varphi\sin\vartheta \\ & - 4\sin^2(\vartheta+\varphi)\sin\vartheta\cos^2\vartheta] + \cdots\end{aligned} \qquad \text{Eqn. (30)}$$

We shall find that the sine and the cosine of angle $(i_1'+\theta')$ will be needed, therefore we shall derive the equations for $\sin(i_1'+\theta')$ and $\cos(i_1'+\theta')$ at this point.

$$\sin(i_1'+\vartheta') = \sin i_1' \cos\vartheta' + \cos i_1' \sin\vartheta'$$

Substitute equations 29 and 30.

$$\begin{aligned}\sin(i_1'+\vartheta') = {} & \cos\vartheta\cos\vartheta' + \frac{v}{c}\sin\vartheta\cos\vartheta'[2\sin(\vartheta+\varphi)\cos\vartheta - \sin\varphi] \\ & + \frac{v^2}{c^2}\cos\vartheta'[-\tfrac{1}{2}\sin^2\varphi\cos\vartheta + 2\sin(\vartheta+\varphi)\sin\varphi - 2\sin^2(\vartheta+\varphi)\cos\vartheta \\ & \qquad -4\sin(\vartheta+\varphi)\sin\varphi\sin^2\vartheta + 4\sin^2(\vartheta+\varphi)\sin^2\vartheta\cos\vartheta] \\ & + \sin\vartheta\sin\vartheta' - \frac{v}{c}\sin\vartheta'\cos\vartheta[2\sin(\vartheta+\varphi)\cos\vartheta - \sin\varphi] \\ & + \frac{v^2}{c^2}\sin\vartheta'[-\tfrac{1}{2}\sin^2\varphi\sin\vartheta + 4\sin(\vartheta+\varphi)\sin\vartheta\cos\vartheta\sin\varphi - 4\sin^2(\vartheta+\varphi)\sin\vartheta\cos^2\vartheta].\end{aligned}$$

Collecting terms and arranging according to powers of v/c, we get:

$$\begin{aligned}\sin(i_1'+\vartheta') = {} & \cos(\vartheta-\vartheta') + \frac{v}{c}\sin(\vartheta-\vartheta')[2\sin(\vartheta+\varphi)\cos\vartheta - \sin\varphi] \\ & + \frac{v^2}{c^2}[-\tfrac{1}{2}\sin^2\varphi\cos(\vartheta-\vartheta') + 2\sin(\vartheta+\varphi)\sin\varphi\cos\vartheta' \\ & \qquad + \sin(\vartheta-\vartheta')\{4\sin^2(\vartheta+\varphi)\sin\vartheta\cos\vartheta - 4\sin(\vartheta+\varphi)\sin\varphi\sin\vartheta\} \\ & \qquad - 2\sin^2(\vartheta+\varphi)\cos\vartheta\cos\vartheta'] + \cdots .\end{aligned}$$

Substituting equations 4, 5, 9, 10, 11, 12, 13 and 14; we obtain:

$$\sin(i_1'+\vartheta') = 1 + \frac{v^2}{c^2}[-\tfrac{1}{2}\sin^2\varphi_0 + 2\sin(\vartheta_0+\varphi_0)\cos\vartheta_0\{\sin\varphi_0 - \sin(\vartheta_0+\varphi_0)\cos\vartheta_0\}] + \cdots \qquad \text{Eqn. (31)}$$

Similarly,

$$\cos(i_1'+\vartheta') = \cos i_1'\cos\vartheta' - \sin i_1'\sin\vartheta'$$

Substitute equations 29 and 30.

$$\begin{aligned}\cos(i_1'+\vartheta') = {} & \sin\vartheta\cos\vartheta' - \frac{v}{c}\cos\vartheta\cos\vartheta'[2\sin(\vartheta+\varphi)\cos\vartheta - \sin\varphi] \\ & + \frac{v^2}{c^2}\cos\vartheta'[-\tfrac{1}{2}\sin^2\varphi\sin\vartheta + 4\sin(\vartheta+\varphi)\sin\vartheta\cos\vartheta(\sin\varphi - \sin(\vartheta+\varphi)\cos\vartheta)] \\ & - \cos\vartheta\sin\vartheta' - \frac{v}{c}\sin\vartheta\sin\vartheta'[2\sin(\vartheta+\varphi)\cos\vartheta - \sin\varphi] \\ & - \frac{v^2}{c^2}\sin\vartheta'[-\tfrac{1}{2}\sin^2\varphi\cos\vartheta + 2\sin(\vartheta+\varphi)\sin\varphi - 2\sin^2(\vartheta+\varphi)\cos\vartheta \\ & \qquad - 4\sin(\vartheta+\varphi)\sin\varphi\sin^2\vartheta + 4\sin^2(\vartheta+\varphi)\sin^2\vartheta\cos\vartheta]\end{aligned}$$

Collecting terms and arranging according to powers of v/c, we get

$$\cos(i_1'+\vartheta') = \sin(\vartheta-\vartheta') - \frac{v}{c}(\cos(\vartheta-\vartheta')[2\sin(\vartheta+\varphi)\cos\vartheta - \sin\varphi]$$
$$+\frac{v^2}{c^2}[-\tfrac{1}{2}\sin^2\varphi\sin(\vartheta-\vartheta') - 2\sin(\vartheta+\varphi)\sin\varphi\sin\vartheta'$$
$$+4\cos(\vartheta-\vartheta')\sin(\vartheta+\varphi)\sin\vartheta\{\sin\varphi - \sin(\vartheta+\varphi)\cos\vartheta\}$$
$$+2\sin^2(\vartheta+\varphi)\cos\vartheta\sin\vartheta'] + \cdots$$

Substituting equations 4, 5, 9, 10, 11, 12, 13 and 14; we obtain:

$$\cos(i_1'+\vartheta') = -\frac{v}{c}[2\sin(\vartheta_0+\varphi_0)\cos\vartheta_0 - \sin\varphi_0]$$
$$+\frac{v^2}{c^2}[\sin\vartheta_0\cos\vartheta_0\{\cos^2\varphi_0 - \cos^2(\vartheta_0+\varphi_0)$$
$$+\tan\vartheta_0(\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0) - \sin\varphi_0\cos\varphi_0)\}$$
$$+2\sin(\vartheta_0+\varphi_0)\sin\vartheta_0(\sin\varphi_0 - \sin(\vartheta_0+\varphi_0)\cos\vartheta_0)] + \cdots$$

The second order term reduces to zero, hence, we get:

$$\cos(i_1'+\vartheta') = -\frac{v}{c}[2\sin(\vartheta_0+\varphi_0)\cos\vartheta_0 - \sin\varphi_0] + \cdots$$
$$= -\frac{v}{c}\sin(2\vartheta_0+\varphi_0) + \cdots \qquad \text{Eqn. (32)}$$

ANGLE OF REFLECTION OF THE RAY FROM MIRROR M'

To find the sine and the cosine of the angle of reflection at M' apply equations 21 and 22. Since the mirror is approaching the source of light, the velocity is positive. The angle between the mirror M' and the direction of the velocity is $(\Theta+\Theta'+\phi+\sigma-\frac{\pi}{2})$, therefore, sin ϕ must be replaced by:

$$\sin(\vartheta+\vartheta'+\varphi+\sigma-\tfrac{\pi}{2}) = -\cos(\vartheta+\vartheta'+\varphi+\sigma).$$

Expanding the latter expression, we get:

$$\sin(\vartheta+\vartheta'+\varphi+\sigma-\tfrac{\pi}{2}) = -\cos(\vartheta+\vartheta'+\varphi)\cos\sigma + \sin(\vartheta+\vartheta'+\varphi)\sin\sigma.$$

Substituting equations 7, 8, 15, and 16 and observing that $(v/c)^1$ is the

highest power to which this expression need be developed, we get:

$$\sin(\vartheta+\vartheta'+\varphi+\sigma-\tfrac{\pi}{2}) = -\cos(2\vartheta_0+\varphi_0)+\cdots.$$

The $(v/c)^1$ term is equal to zero. Hence, substituting in equations 21 and 22, we get:

$$\sin i_2' = \sin i_2 + 2\frac{v}{c}\cos(2\vartheta_0+\varphi_0)\sin i_2\cos i_2 + 2\frac{v^2}{c^2}\cos^2(2\vartheta_0+\varphi_0)\sin i_2(2\cos^2 i_2-1)+\cdots \qquad \text{Eqn. (33)}$$

and

$$\cos i_2' = \cos i_2 - 2\frac{v}{c}\cos(2\vartheta_0+\varphi_0)\sin^2 i_2 - 4\frac{v^2}{c^2}\cos^2(2\vartheta_0+\varphi_0)\sin^2 i_2\cos i_2+\cdots \qquad \text{Eqn. (34)}$$

To find i_2 construct the lines AE and CK in figure I parallel to S'. Since they are bounded by the same parallel lines, they are both equal to S'. Then,

$$\psi = i_1' + \vartheta' - \frac{\pi}{2}.$$

Also,

angle AEK $= \psi = i_2 - \sigma$.

Substituting the latter expression for ψ in the former equation, we get:

$$i_2 = i_1' + \vartheta' + \sigma - \frac{\pi}{2} \qquad \text{Eqn. (35)}$$

Now

$$\sin i_2 = -\cos(i_1'+\vartheta'+\sigma) = -\cos(i_1'+\vartheta')\cos\sigma + \sin(i_1'+\vartheta')\sin\sigma$$

and

$$\cos i_2 = \sin(i_1'+\vartheta'+\sigma) = \sin(i_1'+\vartheta')\cos\sigma + \cos(i_1'+\vartheta')\sin\sigma .$$

Substitute equations 15, 16, . This gives:

$$\sin i_2 = -\cos(i_1'+\vartheta') + \frac{v^2}{c^2}\cos(2\vartheta_0+\varphi_0)\sin(2\vartheta_0+\varphi_0)\sin(i_1'+\vartheta')+\cdots$$

and

$$\cos i_2 = \sin(i_1'+\vartheta') + \frac{v^2}{c^2}\cos(2\vartheta_0+\varphi_0)\sin(2\vartheta_0+\varphi_0)\cos(i_1'+\vartheta')+\cdots .$$

Substitute equations 31 and 32 for $\sin(i_1'+\theta')$ and $\cos(i_1'+\theta')$. This gives:

$$\sin i_2 = \frac{v}{c}\sin(2\vartheta_0+\varphi_0) + \frac{v^2}{c^2}\cos(2\vartheta_0+\varphi_0)\sin(2\vartheta_0+\varphi_0) + \cdots \qquad \text{Eqn. (36)}$$

and

$$\cos i_2 = 1 + \frac{v^2}{c^2}\left[-\tfrac{1}{2}\sin^2\varphi_0 + 2\sin(\vartheta_0+\varphi_0)\cos\vartheta_0\{\sin\varphi_0 - \sin(\vartheta_0+\varphi_0)\cos\vartheta_0\}\right] + \cdots \qquad \text{Eqn. (37)}$$

The following expression will be developed only up to the power of v/c that it will be needed when it is substituted in equation 33.

$$\sin i_2 \cos i_2 = \frac{v}{c}\sin(2\vartheta_0+\varphi_0) + \cdots \qquad \text{Eqn. (38)}$$

Since $\sin i_2$ is of the order of magnitude of v/c, its appearance to the second power in the first order term of $\cos i_2'$ causes that term to be dropped; a fortiori in the second order term. Likewise, its appearance in the second order term of $\sin i_2'$ causes that term to be dropped.

Substituting equations 36 and 38 in equation 33 gives:

$$\sin i_2' = \frac{v}{c}\sin(2\vartheta_0+\varphi_0) + \frac{v^2}{c^2}\cos(2\vartheta_0+\varphi_0)\sin(2\vartheta_0+\varphi_0)$$
$$+ 2\frac{v^2}{c^2}\cos(2\vartheta_0+\varphi_0)\sin(2\vartheta_0+\varphi_0) + \cdots,$$

which, on further simplification, becomes:

$$\sin i_2' = \frac{v}{c}\sin(2\vartheta_0+\varphi_0) + 3\frac{v^2}{c^2}\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0) + \cdots.$$

Eqn. (39)

Also, substituting equation 37 in equation 34, we get:

$$\cos i_2' = 1 + \frac{v^2}{c^2}\left[-\tfrac{1}{2}\sin^2\varphi_0 + 2\sin(\vartheta_0+\varphi_0)\cos\vartheta_0\{\sin\varphi_0 - \sin(\vartheta_0+\varphi_0)\cos\vartheta_0\}\right] + \cdots$$

$$\cos i_2' = 1 - \frac{v^2}{c^2}\left[\tfrac{1}{2}\sin^2\varphi_0 + 2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\right] + \cdots. \qquad \text{Eqn. (40)}$$

We shall find that the sine and the cosine of the angle $(\theta'+i_2'+\sigma)$ will be needed, therefore we shall derive the equations for $\sin(\theta'+i_2'+\sigma)$ and $\cos(\theta'+i_2'+\sigma)$ at this point.

$$\sin(\theta'+i_2'+\sigma) = \sin(\theta'+i_2')\cos\sigma + \cos(\theta'+i_2')\sin\sigma$$

and

$$\cos(\theta'+i_2'+\sigma) = \cos(\theta'+i_2')\cos\sigma - \sin(\theta'+i_2')\sin\sigma \; .$$

Substitute equations 15 and 16 for $\sin\sigma$ and $\cos\sigma$.

$$\sin(\theta'+i_2'+\sigma) = \sin(\theta'+i_2') + \frac{v^2}{c^2}\sin(2\theta_0+\varphi_0)\cos(2\theta_0+\varphi_0)\cos(\theta'+i_2') + \cdots \qquad \text{Eqn. (41)}$$

and

$$\cos(\theta'+i_2'+\sigma) = \cos(\theta'+i_2') - \frac{v^2}{c^2}\sin(2\theta_0+\varphi_0)\cos(2\theta_0+\varphi_0)\sin(\theta'+i_2') + \cdots \qquad \text{Eqn. (42)}$$

Expand $\sin(\theta'+i_2')$.

$$\sin(\theta'+i_2') = \sin\theta'\cos i_2' + \cos\theta'\sin i_2'$$

Substitute equations 11, 12, 39 and 40.

$$\begin{aligned}\sin(\theta'+i_2') = {} & \sin\theta_0\left[1 - \tfrac{1}{2}\tfrac{v^2}{c^2}\{\sin^2(2\theta_0+\varphi_0) - \cos^2(\theta_0+\varphi_0)\}\right] \\ & \left[1 - \tfrac{v^2}{c^2}\{\tfrac{1}{2}\sin^2\varphi_0 + 2\sin(\theta_0+\varphi_0)\cos(\theta_0+\varphi_0)\sin\theta_0\cos\theta_0\}\right] \\ & + \cos\theta_0\left[\tfrac{v}{c}\sin(2\theta_0+\varphi_0) + 3\tfrac{v^2}{c^2}\sin(2\theta_0+\varphi_0)\cos(2\theta_0+\varphi_0)\right] + \cdots\end{aligned}$$

Multiply and arrange according to powers of v/c.

$$\begin{aligned}\sin(\theta'+i_2') = {} & \sin\theta_0 + \tfrac{v}{c}\sin(2\theta_0+\varphi_0)\cos\theta_0 \\ & + \tfrac{v^2}{c^2}\left[-\tfrac{1}{2}\sin^2(2\theta_0+\varphi_0)\sin\theta_0 + \tfrac{1}{2}\cos^2(\theta_0+\varphi_0)\sin\theta_0 - \tfrac{1}{2}\sin^2\varphi_0\sin\theta_0\right. \\ & - 2\sin(\theta_0+\varphi_0)\cos(\theta_0+\varphi_0)\sin^2\theta_0\cos\theta_0 \\ & \left. + 3\sin(2\theta_0+\varphi_0)\cos(2\theta_0+\varphi_0)\cos\theta_0\right] \; .\end{aligned}$$

Similarly,

$$\cos(\vartheta'+i_2') = \cos\vartheta'\cos i_2' - \sin\vartheta'\sin i_2'$$

Again, substituting equations 11, 12, 39 and 40; we get:

$$\cos(\vartheta'+i_2') = \cos\vartheta_0\left[1+\tfrac{1}{2}\tfrac{v^2}{c^2}\{\cos^2(2\vartheta_0+\varphi_0)+\cos(\vartheta_0+\varphi_0)[2\tan\vartheta_0\sin(\vartheta_0+\varphi_0)-\cos(\vartheta_0+\varphi_0)]\}\right]$$
$$\left[1-\tfrac{v^2}{c^2}\{\tfrac{1}{2}\sin^2\varphi_0+2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\}\right]$$
$$-\sin\vartheta_0\left[\tfrac{v}{c}\sin(2\vartheta_0+\varphi_0)+3\tfrac{v^2}{c^2}\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\right]+\cdots.$$

Multiply and arrange according to powers of v/c.

$$\cos(\vartheta'+i_2') = \cos\vartheta_0 - \tfrac{v}{c}\sin(2\vartheta_0+\varphi_0)\sin\vartheta_0$$
$$+\tfrac{v^2}{c^2}\Big[\tfrac{1}{2}\cos^2(2\vartheta_0+\varphi_0)\cos\vartheta_0+\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0$$
$$-\tfrac{1}{2}\cos^2(\vartheta_0+\varphi_0)\cos\vartheta_0-\tfrac{1}{2}\sin^2\varphi_0\cos\vartheta_0$$
$$-2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\cos^2\vartheta_0$$
$$-3\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\sin\vartheta_0\Big]+\cdots.$$

Substituting for $\sin(\theta'+i_2')$ and $\cos(\theta'+i_2')$ in equation 41, we get:

$$\sin(\vartheta'+i_2'+\sigma) = \sin\vartheta_0+\tfrac{v}{c}\sin(2\vartheta_0+\varphi_0)\cos\vartheta_0$$
$$+\tfrac{v^2}{c^2}\Big[-\tfrac{1}{2}\sin^2(2\vartheta_0+\varphi_0)\sin\vartheta_0+\tfrac{1}{2}\cos^2(\vartheta_0+\varphi_0)\sin\vartheta_0$$
$$-\tfrac{1}{2}\sin^2\varphi_0\sin\vartheta_0$$
$$-2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin^2\vartheta_0\cos\vartheta_0$$
$$+4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\cos\vartheta_0\Big]+\cdots. \quad \text{Eqn. (43)}$$

For later use, it will be convenient to write this equation symbolically as:

$$\sin(\vartheta'+i_2'+\sigma) = E+\tfrac{v}{c}F+\tfrac{v^2}{c^2}P+\cdots. \quad \text{Eqn. (44)}$$

where E, F and P are defined as the coefficients of the $(v/c)^0$, $(v/c)^1$ and $(v/c)^2$ terms respectively.

Similarly, making the proper substitutions in equation 42, we get:

$$\cos(\vartheta'+i_2'+\sigma) = \cos\vartheta_0 - \frac{v}{c}\sin(2\vartheta_0+\varphi_0)\sin\vartheta_0$$
$$+\frac{v^2}{c^2}\Big[\tfrac{1}{2}\cos^2(2\vartheta_0+\varphi_0)\cos\vartheta_0 + \sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0$$
$$-\tfrac{1}{2}\cos^2(\vartheta_0+\varphi_0)\cos\vartheta_0 - \tfrac{1}{2}\sin^2\varphi_0\cos\vartheta_0$$
$$-2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\cos^2\vartheta_0$$
$$-4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\sin\vartheta_0\Big]+\cdots \,. \qquad \text{Eqn. (45)}$$

Also, we shall find that the reciprocal of $\sin(\theta'+i_2'+\sigma)$ will be needed up to the first power of v/c.

$$\frac{1}{\sin(\vartheta'+i_2'+\sigma)} = \frac{1}{\sin\vartheta_0} - \frac{v}{c}\frac{\cos\vartheta_0}{\sin^2\vartheta_0}\sin(2\vartheta_0+\varphi_0)+\cdots \,. \qquad \text{Eqn. (46)}$$

ANGLE OF REFLECTION OF THE RAY FROM THE MIRROR M"

To find the sine and the cosine of the angle of reflection at mirror M" apply equations 21 and 22. Since the mirror is receding from the source of light, the velocity is negative. The angle between M" and the direction of the velocity is $(\frac{\pi}{2}-\phi+\omega)$, therefore, $\sin\phi$ must be replaced by $\sin(\frac{\pi}{2}-\phi+\omega)$ or $\cos(\phi-\omega)$. Hence:

$$\sin j_1' = \sin j_1 + 2\frac{v}{c}\cos(\varphi-\omega)\sin j_1\cos j_1$$
$$+2\frac{v^2}{c^2}\cos^2(\varphi-\omega)\sin j_1(2\cos^2 j_1-1)+\cdots \,, \qquad \text{Eqn. (47)}$$

and

$$\cos j_1' = \cos j_1 - 2\frac{v}{c}\cos(\varphi-\omega)\sin^2 j_1$$
$$-4\frac{v^2}{c^2}\cos^2(\varphi-\omega)\sin^2 j_1\cos j_1+\cdots \,. \qquad \text{Eqn. (48)}$$

Now

$$\cos(\varphi-\omega) = \cos\varphi\cos\omega + \sin\varphi\sin\omega$$

Substituting equations 3, 4, 17 and 18 and observing that $\cos(\phi-\omega)$ need be developed only up to the first power of v/c, we obtain:

$$\cos(\varphi-\omega) = \cos\varphi_0 + \frac{v}{c}\sin\varphi_0\sin\mu_0 + \cdots .$$

Also:

$$\cos^2(\varphi-\omega) = \cos^2\varphi_0 + \cdots$$

to the order of approximation to which it is needed.

Referring to figure I we see that:

$$j_1 = \alpha - \omega$$

Therefore,

$$\sin j_1 = \sin(\alpha-\omega)$$
$$= \sin\alpha\cos\omega - \cos\alpha\sin\omega$$

Substituting equations 17, 18, 24 and 25 gives:

$$\sin j_1 = \frac{v}{c}(\sin\varphi_0 - \sin\mu_0) - \frac{v^2}{c^2}\sin\varphi_0\cos\varphi_0 + \cdots ,$$

Similarly:

$$\cos j_1 = \cos(\alpha-\omega)$$
$$= \cos\alpha\cos\omega + \sin\alpha\sin\omega$$

Making the same substitutions as in the previous case, we get:

$$\cos j_1 = 1 - \frac{1}{2}\frac{v^2}{c^2}[\sin^2\varphi_0 + \sin^2\mu_0 - 2\sin\varphi_0\sin\mu_0] + \cdots$$

The following expression will be developed only up to the power of

v/c to which it will be needed when it is substituted in equation 41.

$$\sin j_1 \cos j_1 = \frac{v}{c}(\sin\varphi_0 - \sin\mu_0) + \cdots$$

The remark made concerning sin i_2, on page 43, may likewise be made about sin j_1. Hence, making the proper substitutions in equations 47 and 48, we get:

$$\sin j_1' = \frac{v}{c}(\sin\varphi_0 - \sin\mu_0) + \frac{v^2}{c^2}\cos\varphi_0(\sin\varphi_0 - 2\sin\mu_0) + \ldots \qquad \text{Eqn. (49)}$$

and

$$\cos j_1' = 1 - \frac{1}{2}\frac{v^2}{c^2}(\sin^2\varphi_0 + \sin^2\mu_0 - 2\sin\varphi_0\sin\mu_0) + \cdots. \qquad \text{Eqn. (50)}$$

We shall find that the sine and the cosine of the angle ($\theta - j_1'$) will be needed, therefore, we shall derive the equations for sin($\theta - j_1'$) and cos($\theta - j_1'$) at this point.

$$\sin(\vartheta - j_1') = \sin\vartheta\cos j_1' - \cos\vartheta\sin j_1'$$

and

$$\cos(\vartheta - j_1') = \cos\vartheta\cos j_1' + \sin\vartheta\sin j_1'.$$

Substituting equations 9, 10, 49 and 50 in the former expression, gives:

$$\sin(\vartheta - j_1') = \sin\vartheta_0[1 - \frac{1}{2}\frac{v^2}{c^2}\{\sin^2(\vartheta_0+\varphi_0) - \cos^2\varphi_0\}]$$
$$[1 - \frac{1}{2}\frac{v^2}{c^2}\{\sin^2\varphi_0 + \sin^2\mu_0 - 2\sin\varphi_0\sin\mu_0\}]$$
$$- \cos\vartheta_0[\frac{v}{c}\{\sin\varphi_0 - \sin\mu_0\} + \frac{v^2}{c^2}\cos\varphi_0\{\sin\varphi_0 - 2\sin\mu_0\}] + \cdots.$$

Multiply and arrange according to powers of v/c.

$$\sin(\vartheta - j_1') = \sin\vartheta_0 - \frac{v}{c}\cos\vartheta_0[\sin\varphi_0 - \sin\mu_0]$$
$$+ \frac{v^2}{c^2}[\frac{1}{2}\sin\vartheta_0\{\sin^2(\vartheta_0+\varphi_0) - \cos^2\varphi_0 + \sin^2\varphi_0 + \sin^2\mu_0 - 2\sin\varphi_0\sin\mu_0\}$$
$$+ \cos\vartheta_0\cos\varphi_0\{\sin\varphi_0 - 2\sin\mu_0\}] + \cdots \qquad \text{Eqn. (51)}$$

Again, making the same substitutions in the expression for $\cos(\theta - j_1')$, we get:

$$\cos(\vartheta - j_1') = \cos\vartheta_0\left[1 + \frac{1}{2}\frac{v^2}{c^2}\{\cos^2(\vartheta_0+\varphi_0) + \cos\varphi_0(2\tan\vartheta_0\sin\varphi_0 - \cos\varphi_0)\}\right]$$
$$\left[1 - \frac{1}{2}\frac{v^2}{c^2}\{\sin^2\varphi_0 + \sin^2\mu_0 - 2\sin\varphi_0\sin\mu_0\}\right]$$
$$+ \sin\vartheta_0\left[\frac{v}{c}(\sin\varphi_0 - \sin\mu_0) + \frac{v^2}{c^2}\cos\varphi_0(\sin\varphi_0 - 2\sin\mu_0)\right] + \cdots.$$

Multiply and arrange according to powers v/c.

$$\cos(\vartheta - j_1') = \cos\vartheta_0 + \frac{v}{c}\sin\vartheta_0(\sin\varphi_0 - \sin\mu_0)$$
$$+ \frac{v^2}{c^2}\left[\frac{1}{2}\cos^2(\vartheta_0+\varphi_0) + 2\sin\vartheta_0\sin\varphi_0\cos\varphi_0 - \frac{1}{2}\cos\vartheta_0\right.$$
$$+ \cos\vartheta_0\sin\varphi_0\sin\mu_0 - 2\sin\vartheta_0\cos\varphi_0\sin\mu_0$$
$$\left. - \frac{1}{2}\cos\vartheta_0\sin^2\mu_0\right] + \cdots, \qquad \text{Eqn. (52)}$$

ANGLE OF REFLECTION FROM PLATE M AFTER RETURN FROM MIRROR M"

To find the sine and the cosine of the angle of reflection at the plate M after the ray has been reflected from M" apply equations 21 and 22. Since the mirror is approaching the source of light, the velocity is positive. The angle between the plate M and the direction of the velocity is $(\theta + \phi)$, therefore, substituting $(\theta + \phi)$ for ϕ, we get:

$$\sin j_2' = \sin j_2 - 2\frac{v}{c}\sin(\vartheta+\varphi)\sin j_2\cos j_2$$
$$+ 2\frac{v^2}{c^2}\sin^2(\vartheta+\varphi)\sin j_2(2\cos^2 j_2 - 1) + \cdots, \qquad \text{Eqn. (53)}$$

and

$$\cos j_2' = \cos j_2 + 2\frac{v}{c}\sin(\vartheta+\varphi)\sin^2 j_2$$
$$- 4\frac{v^2}{c^2}\sin^2(\vartheta+\varphi)\sin^2 j_2\cos j_2 + \cdots.$$

Eqn. (54)

Referring to figure I, we observe that:

$$j_2 = \frac{\pi}{2} - \vartheta - \omega + j_1' .$$

Therefore,

$$\sin j_2 = \cos(\vartheta + \omega - j_1')$$

$$= \cos(\vartheta - j_1')\cos\omega - \sin(\vartheta - j_1')\sin\omega$$

and

$$\cos j_2 = \sin(\vartheta + \omega - j_1')$$

$$= \sin(\vartheta - j_1')\cos\omega + \cos(\vartheta - j_1')\sin\omega .$$

Substituting equations 17 and 18 in expression for sin j_2, we get:

$$\sin j_2 = \cos(\vartheta - j_1') - \frac{v}{c}\sin\mu_0 \sin(\vartheta - j_1')$$
$$- \frac{v^2}{c^2}\left[\frac{1}{2}\sin^2\mu_0\cos(\vartheta - j_1') + \sin\varphi_0\cos\varphi_0\sin(\vartheta - j_1')\right] + \cdots .$$

Now, replacing sin(θ-j_1') and cos(θ-j_1') by equations 51 and 52, respectively, we get:

$$\sin j_2 = \cos\vartheta_0 + \frac{v}{c}\sin\vartheta_0[\sin\varphi_0 - 2\sin\mu_0]$$
$$+ \frac{v^2}{c^2}\left[\frac{1}{2}\cos^2(\vartheta_0+\varphi_0)\cos\vartheta_0 + \sin\vartheta_0\sin\varphi_0\cos\varphi_0\right.$$
$$- \frac{1}{2}\cos\vartheta_0 + 2\cos\vartheta_0\sin\varphi_0\sin\mu_0$$
$$\left. - 2\sin\vartheta_0\cos\varphi_0\sin\mu_0 - 2\cos\vartheta_0\sin^2\mu_0\right] + \cdots .$$

Likewise, substituting equations 17 and 18 in expression for cos j_2, we get:

$$\cos j_2 = \sin(\vartheta - j_1') + \frac{v}{c}\sin\mu_0\cos(\vartheta - j_1')$$
$$- \frac{v^2}{c^2}\left[\frac{1}{2}\sin^2\mu_0\sin(\vartheta - j_1') - \sin\varphi_0\cos\varphi_0\cos(\vartheta - j_1')\right] + \cdots .$$

Again, replacing sin(θ-j_1') and cos(θ-j_1') by equations 51 and 52, respectively,

we get:

$$\cos j_2 = \sin \vartheta_0 - \frac{v}{c}\cos\vartheta_0[\sin\varphi_0 - 2\sin\mu_0]$$
$$+\frac{v^2}{c^2}[-\tfrac{1}{2}\sin^2(\vartheta_0+\varphi_0)\sin\vartheta_0 + \tfrac{1}{2}\sin\vartheta_0\cos^2\varphi_0$$
$$-\frac{1}{2}\sin\vartheta_0\sin^2\varphi_0 + 2\sin\vartheta_0\sin\varphi_0\sin\mu_0$$
$$+2\cos\vartheta_0\cos\varphi_0\sin\mu_0 - 2\sin\vartheta_0\sin^2\mu_0]+\cdots.$$

The following expressions will be developed only up to the powers of v/c to which they are needed when substituted in equations 53 and 54.

$$\sin j_2 \cos j_2 = \sin\vartheta_0\cos\vartheta_0 + \frac{v}{c}(1-2\cos^2\vartheta_0)(\sin\varphi_0 - 2\sin\mu_0)+\cdots$$

$$\sin^2 j_2 = \cos^2\vartheta_0 + 2\frac{v}{c}\sin\vartheta_0\cos\vartheta_0(\sin\varphi_0 - 2\sin\mu_0)+\cdots$$

$$\sin j_2\cos^2 j_2 = \sin^2\vartheta_0\cos\vartheta_0+\cdots$$

$$\sin^2 j_2\cos j_2 = \sin\vartheta_0\cos^2\vartheta_0+\cdots$$

Substituting the appropriate expressions in equation 53, we obtain:

$$\sin j_2' = \cos\vartheta_0 - \frac{v}{c}\sin\vartheta_0[\sin(2\vartheta_0+\varphi_0)+2\sin\mu_0]$$
$$+\frac{v^2}{c^2}[\tfrac{1}{2}\cos^2(\vartheta_0+\varphi_0)\cos\vartheta_0 + \sin\vartheta_0\sin\varphi_0\cos\varphi_0 - \tfrac{1}{2}\cos\vartheta_0$$
$$-2\sin(\vartheta_0+\varphi_0)\sin\varphi_0 + 4\sin(\vartheta_0+\varphi_0)\cos^2\vartheta_0\sin\varphi_0$$
$$+4\sin^2(\vartheta_0+\varphi_0)\sin^2\vartheta_0\cos\vartheta_0 - 2\sin^2(\vartheta_0+\varphi_0)\cos\vartheta_0$$
$$+2\cos\vartheta_0\sin\varphi_0\sin\mu_0 - 2\sin\vartheta_0\cos\varphi_0\sin\mu_0$$
$$+4\sin(\vartheta_0+\varphi_0)\sin\mu_0 - 8\sin(\vartheta_0+\varphi_0)\cos^2\vartheta_0\sin\mu_0$$
$$-2\cos\vartheta_0\sin^2\mu_0]+\cdots.$$

Eqn. (55)

Also, inserting the proper expressions in equation 54, we get:

$$\cos j_2' = \sin\vartheta_0 + \frac{v}{c}\cos\vartheta_0[\sin(2\vartheta_0+\varphi_0) + 2\sin\mu_0]$$
$$+\frac{v^2}{c^2}[-\tfrac{1}{2}\sin^2(\vartheta_0+\varphi_0)\sin\vartheta_0 + \tfrac{1}{2}\sin\vartheta_0\cos^2\varphi_0$$
$$-\tfrac{1}{2}\sin\vartheta_0\sin^2\varphi_0 + 4\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\sin\varphi_0$$
$$-4\sin^2(\vartheta_0+\varphi_0)\sin\vartheta_0\cos^2\vartheta_0 + 2\sin\vartheta_0\sin\varphi_0\sin\mu_0$$
$$+2\cos\vartheta_0\cos\varphi_0\sin\mu_0$$
$$-8\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\sin\mu_0$$
$$-2\sin\vartheta_0\sin^2\mu_0]+\cdots.$$

Eqn. (56)

For later use, it will be convenient to write this equation symbolically, as:

$$\cos j_2' = R + \frac{v}{c}S + \frac{v^2}{c^2}T + \cdots.$$

Eqn. (57)

where R, S and T are defined as the coefficient of the $(v/c)^0$, $(v/c)^1$ and $(v/c)^2$ term, respectively.

ANGLE BETWEEN THE TWO FINAL RAYS

Referring to figure I we see that the angle between the ray reflected from M' and the normal to the plate M at the point H is i_3 or $(\frac{\pi}{2}-\vartheta'-\sigma-i_2')$. As a proof of this latter statement, we may sum the angles in the triangle ACH and set them equal to π.

$$\pi = (\vartheta+\vartheta'+\varphi-\psi) + (i_2+i_2') + (i_3+\frac{\pi}{2}-\vartheta-\varphi)$$

After replacing ψ by its equal $(i_2 - \sigma)$, we obtain:

$$i_3 = \left(\frac{\pi}{2} - \vartheta' - \sigma - i_2'\right).$$

Again referring to figure I we observe that the angles i_3 and j_2' are measured in terms of their angular elevation from parallel base lines--the normals to plate M. Let us then construct a triangle with base N and the angles j_2' and i_3 as shown in figure VII.

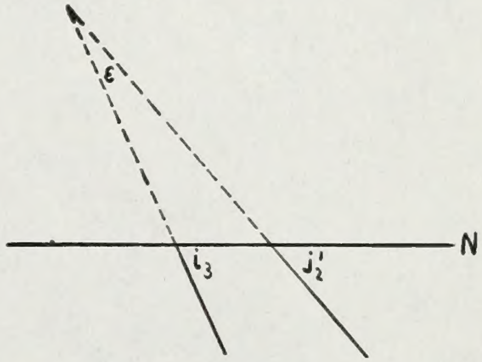

Fig. VII

Unless the two final rays are parallel, there will be a small angle ε at the summit of the triangle. The two base angles of this triangle are $(\pi - i_3)$ and j_2'. Therefore;

$$\varepsilon = \pi - (\pi - i_3 + j_2')$$
$$= i_3 - j_2'.$$

Replacing i_3 by its value found above, we get:

$$\varepsilon = \left(\frac{\pi}{2} - \vartheta' - \sigma - i_2' - j_2'\right)$$

Therefore;

$$\sin\varepsilon = \cos(\vartheta' + \sigma + i_2' + j_2')$$

(For details of the derivation of the following equation see appendix A.)

$$\sin\varepsilon = 2\frac{v}{c}\sin\mu_0 + \frac{v^2}{c^2}\Big[\tfrac{1}{2}\sin\vartheta_0\cos\vartheta_0\{\sin^2(2\vartheta_0+\varphi_0) - \sin^2(\vartheta_0+\varphi_0)\}$$
$$-4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)$$
$$+2\cos(2\vartheta_0+\varphi_0)\sin\mu_0\Big] + \cdots \qquad \text{Eqn. (58)}$$

For later use, it will be convenient to write this equation symbolically, as:

$$\sin\varepsilon = \frac{v}{c}J' + \frac{v^2}{c^2}K' + \frac{v^3}{c^3}M' + \cdots .$$

where J' and K' represent the coefficients of the first and second order terms respectively. It will be found that the third order term, represented by M', will apparently be needed, but upon further development of the problem it will be shown to cancel out completely, hence, it will not be necessary to derive it.

Let us now define ε' by the equation:

$$\sin\varepsilon = \frac{v}{c}\sin\varepsilon'$$

Then,

$$\sin\varepsilon' = J' + \frac{v}{c}K' + \frac{v^2}{c^2}M' + \cdots .$$

We shall find that the reciprocal of $\sin\varepsilon'$ will be needed. Thus;

$$\begin{aligned}\frac{1}{\sin\varepsilon'} = {} & \frac{1}{2\sin\mu_0} \\ & -\frac{v}{c}\frac{1}{4\sin^2\mu_0}\Big[\tfrac{1}{2}\sin\vartheta_0\cos\vartheta_0\{\sin^2(2\vartheta_0+\varphi_0) - \sin^2(\vartheta_0+\varphi_0)\} \\ & \qquad - 4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0) \\ & \qquad + 2\cos(2\vartheta_0+\varphi_0)\sin\mu_0\Big] \\ & + \text{second order terms} + \cdots \end{aligned}\qquad \text{Eqn. (59)}$$

Write this symbolically, as:

$$\frac{1}{\sin\varepsilon'} = J + \frac{v}{c}K + \frac{v^2}{c^2}M + \cdots \qquad \text{Eqn. (60)}$$

where J, K and M represent the coefficient of the $(v/c)^0$, $(v/c)^1$ and $(v/c)^2$ terms, respectively; M not being actually determined.

LENGTH OF PATH FROM PLATE TO MIRROR M'

Referring to figure I we observe that during the interval of time t_1' that the ray reflected at the plate M travels from A to C, the mirror M' has moved from E to C. Let AC = $ct_1' = D_1'$ and $EC = AK = vt_1'$. Then, in triangle ACE, we have, by the sine law: (cf. Appendix B)

$$\frac{\sin[\pi-(\vartheta+\vartheta'+\varphi)]}{D_1'} = \frac{\sin(\vartheta+\vartheta'+\varphi+\sigma-i_2)}{S'}$$

Solving for D_1' ,we obtain:

$$D_1' = S'\frac{\sin(\vartheta+\vartheta'+\varphi)}{\sin(\vartheta+\vartheta'+\varphi+\sigma-i_2)}$$

Expand denominator in terms of angles $(\theta+\theta'+\phi)$ and $(\sigma-i_2)$. Also, divide numerator and denominator by $\sin(\theta+\theta'+\phi)$. This yields:

$$D_1' = S'\frac{1}{\cos(\sigma-i_2) + \dfrac{\cos(\vartheta+\vartheta'+\varphi)\sin(\sigma-i_2)}{\sin(\vartheta+\vartheta'+\varphi)}}$$

Again, in figure I, referring to triangle ACK, we have by the sine law:

$$\frac{\sin(i_2-\sigma)}{v\,t_1'} = \frac{\sin[\pi-(\vartheta+\vartheta'+\varphi)]}{c\,t_1'}$$

Solving for v/c and changing $\sin(i_2-\sigma)$ to its equal $[-\sin(\sigma-i_2)]$ we get:

$$\frac{v}{c} = -\frac{\sin(\sigma - i_2)}{\sin(\vartheta+\vartheta'+\varphi)}$$

Subsitute the latter expression in the equation for D_1'. Also replace i_2 by its equal given in equation 35. This gives:

$$D_1' = S' \frac{1}{\sin(\vartheta'+i_1') - \frac{v}{c}\cos(\vartheta+\vartheta'+\varphi)}$$

Replacing $\sin(\theta'+i_1')$ and $\cos(\theta+\theta'+\phi)$ by equations 31 and 8 respectively, results in:

$$D_1' = S' \frac{1}{1 - \frac{v}{c}\cos(2\vartheta_0+\varphi_0) - \frac{v^2}{c^2}\left[\frac{1}{2}\sin^2\varphi_0 - 2\sin(\vartheta_0+\varphi_0)\cos\vartheta_0\{\sin\varphi_0 - \sin(\vartheta_0+\varphi_0)\cos\vartheta_0\}\right]}$$

or

$$D_1' = S' \frac{1}{1 - \frac{v}{c}\cos(2\vartheta_0+\varphi_0) - \frac{v^2}{c^2}\left[\frac{1}{2}\sin^2\varphi_0 + 2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\right]}$$

Expand this expression in terms of powers of v/c and drop powers greater than the second.

$$D_1' = S'\Big[1 + \frac{v}{c}\cos(2\vartheta_0+\varphi_0) + \frac{v^2}{c^2}\{\cos^2(2\vartheta_0+\varphi_0) + \frac{1}{2}\sin^2\varphi_0 + 2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\} + \cdots\Big]$$

Applying the Lorentz transformation equation 19 to the above, we obtain:

$$D_1' = S_0'\Big[1 + \frac{v}{c}\cos(2\vartheta_0+\varphi_0) + \frac{v^2}{c^2}\{\frac{1}{2}\cos^2(2\vartheta_0+\varphi_0) + \frac{1}{2}\sin^2\varphi_0 + 2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\} + \cdots\Big]$$

Expanding the term $\frac{1}{2}\cos^2(2\theta_0+\phi_0)$ in terms of angles $(\theta_0+\phi_0)$ and θ_0 we find that

the second order term reduces to $\frac{1}{2}$. Thus, we get as the final expression for D_1' :

$$D_1' = S_0'\left[1+\frac{v}{c}\cos(2\vartheta_0+\varphi_0)+\frac{1}{2}\frac{v^2}{c^2}+\cdots\right] \qquad \text{Eqn. (61)}$$

LENGTH OF PATH FROM MIRROR M' BACK TO PLATE

Referring to figure I we observe that in the interval of time t_2' that the ray reflected at the mirror M' travels from C to H , the mirror M' has moved a distance equal to KH. Let $CH = ct_2' = D_2'$ and $KH = vt_2'$. Then, in triangle CHK, we have, by the sine law: (cf. Appendix B)

$$\frac{\sin(\vartheta+\vartheta'+\varphi)}{D_2'} = \frac{\sin(\vartheta+\vartheta'+\varphi+\sigma+i_2')}{S'}$$

As a proof of this latter statement observe that

angle $CHK = i_3+\frac{\pi}{2}-\theta-\phi$

Replace i_3 by its equal $(\frac{\pi}{2}-\theta'-\sigma-i_2')$. Then:

angle $CHK = \pi-\theta'-\sigma-i_2'-\theta-\phi$

The sine of the angle $CHK = \sin[\pi-(\theta+\theta'+\phi+\sigma+i_2')]$

$$= \sin(\vartheta+\vartheta'+\varphi+\sigma+i_2')$$

Return to the equation involving D_2' and solve for D_2'.

$$D_2' = S' \frac{\sin(\vartheta+\vartheta'+\varphi)}{\sin(\vartheta+\vartheta'+\varphi+\sigma+i_2')} .$$

Expand denominator in terms of angles $(\theta+\theta'+\phi)$ and $(\sigma+i_2')$; and divide numerator and denominator by $\sin(\theta+\theta'+\phi)$. This yields:

$$D_2' = S' \frac{1}{\cos(\sigma+i_2') + \dfrac{\cos(\vartheta+\vartheta'+\varphi)\sin(\sigma+i_2')}{\sin(\vartheta+\vartheta'+\varphi)}} .$$

Again, in figure I, referring to triangle CHK we have by the sine law:

$$\frac{v t_2'}{c t_2'} = \frac{\sin(i_2'+\sigma)}{\sin(\vartheta+\vartheta'+\varphi)} .$$

Substitute the latter expression in the equation for D_2'. This gives:

$$D_2' = S' \frac{1}{\cos(\sigma+i_2') + \frac{v}{c}\cos(\vartheta+\vartheta'+\varphi)} .$$

Expand $\cos(\sigma+i_2')$ and replace $\cos\sigma$ and $\sin\sigma$ by equations 15 and 16 respectively. This results in the equation:

$$D_2' = S' \frac{1}{\cos i_2' + \frac{v}{c}\cos(\vartheta+\vartheta'+\varphi) - \frac{v^2}{c^2}\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\sin i_2'} .$$

Substituting equations 8, 39 and 40 and dropping powers of v/c greater than the second, gives:

$$D_2' = S' \frac{1}{1+\frac{v}{c}\cos(2\vartheta_0+\varphi_0) - \frac{v^2}{c^2}\left[\frac{1}{2}\sin^2\varphi_0 + 2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\right]} .$$

Expand in terms of powers of v/c.

$$D_2' = S'\left[1 - \frac{v}{c}\cos(2\vartheta_0+\varphi_0) + \frac{v^2}{c^2}\left\{\cos^2(2\vartheta_0+\varphi_0) + \frac{1}{2}\sin^2\varphi_0 + 2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\right\} + \cdots\right].$$

Applying the Lorentz transformation equation 19 to the above, we obtain:

$$D_2' = S_0'\left[1 - \frac{v}{c}\cos(2\vartheta_0+\varphi_0) + \frac{v^2}{c^2}\left\{\frac{1}{2}\cos^2(2\vartheta_0+\varphi_0) + \frac{1}{2}\sin^2\varphi_0 + 2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\right\} + \cdots\right]$$

Reducing the second order term as we did in the case of D_1' we obtain as the final expression for D_2' :

$$D_2' = S_0'\left[1 - \frac{v}{c}\cos(2\vartheta_0+\varphi_0) + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right] \qquad \text{Eqn. (62)}$$

It will be observed, that except for the sign of the v/c terms the two distances D_1' and D_2' are equal.

LENGTH OF PATH FROM PLATE M TO MIRROR M"

In figure I construct the line AQ parallel to S". This line will also equal S" since they are both bounded by the same parallel lines. During the interval of time that the ray has traveled from A to P the mirror M" has moved from Q to P . Let $AP = ct_1'' = D_1''$ and $QP = vt_1''$. Then, in triangle APQ we have by the sine law:

$$\frac{\sin(\varphi-\alpha)}{S''} = \frac{\sin(\pi-\varphi)}{D_1''}$$

Solving for D_1'' we obtain:

$$D_1'' = S'' \frac{\sin\varphi}{\sin(\varphi-\alpha)}$$

Expand the denominator.

$$D_1'' = S'' \frac{\sin\varphi}{\sin\varphi\cos\alpha - \cos\varphi\sin\alpha}$$

Substitute equations 3, 4, 24 and 25 and divide numerator and denominator by expression for sin ϕ. This yields,

$$D_1'' = S'' \frac{1}{1-\frac{v}{c}\cos\varphi_0 - \frac{1}{2}\frac{v^2}{c^2}\sin^2\varphi_0}$$

Expanding in terms of powers of v/c we obtain

$$D_1'' = S''\left[1+\frac{v}{c}\cos\varphi_0 + \frac{1}{2}\frac{v^2}{c^2}(1+\cos^2\varphi_0)+\cdots\right]$$

Applying the Lorentz transformation equation 20 we obtain as the final equation for D_1'' :

$$D_1'' = S_0''\left[1+\frac{v}{c}\cos\varphi_0 + \frac{1}{2}\frac{v^2}{c^2}+\cdots\right] \qquad \text{Eqn. (63)}$$

LENGTH OF PATH FROM MIRROR M" BACK TO PLATE M

In figure I construct FP parallel to S" . This line will also equal S" since they are both bounded by the same parallel lines. During the interval of time that the ray of light has traveled from P to L the plate M has travelled a distance FR. Let PL= $ct_2'' = D_2''$ and FR = vt_2''. Let B be the point of intersection of the plate M and the line FP at the time t_2'' .

Now, $$FB + BP = S''$$

The distance that the plate M has moved in the interval of time t_2'' is FN or $vt_2'' \cos[\frac{\pi}{2}-(\theta+\phi)]$ or $vt_2'' \sin(\theta+\phi)$. The latter is the perpendicular distance from F to M at the end of the time interval t_2''. Therefore, in triangle FNB

$$FB = \frac{vt_2'' \sin(\vartheta+\varphi)}{\sin\vartheta}.$$

Substituting this value of FB in the equation above, we obtain:

$$BP = S'' - \frac{vt_2'' \sin(\vartheta+\varphi)}{\sin\vartheta}.$$

In triangle BPL, we have, by the sine law:

$$\frac{\sin(\pi-\vartheta)}{ct_2''} = \frac{\sin(\frac{\pi}{2}-j_2)}{BP}.$$

Replacing BP by its value just obtained and solving for t_2'', we obtain:

$$t_2'' = S'' \frac{\sin\vartheta}{c\cos j_2 + v\sin(\vartheta+\varphi)}.$$

Multiplying both sides of the equation by c and replacing ct_2'' by its equal, D_2'', yields:

$$D_2'' = S'' \frac{\sin\vartheta}{\cos j_2 + \frac{v}{c}\sin(\vartheta+\varphi)}.$$

Substituting equation 5 for $\sin(\theta+\phi)$ and the equation given in page 51, for $\cos j_2$ we obtain as the denominatot of the expression for D_2'' :

$$\cos j_2 + \frac{v}{c}\sin(\vartheta+\varphi) = \sin\vartheta_0 + \frac{v}{c}[\sin\vartheta_0\cos\varphi_0 + 2\cos\vartheta_0\sin\mu_0]$$
$$+\frac{v^2}{c^2}[-\tfrac{1}{2}\sin^2(\vartheta_0+\varphi_0)\sin\vartheta_0 + \tfrac{1}{2}\sin\vartheta_0\cos^2\varphi_0$$
$$\tfrac{1}{2}\sin\vartheta_0\sin^2\varphi_0 + 2\sin\vartheta_0\sin\varphi_0\sin\mu_0$$
$$+2\cos\vartheta_0\cos\varphi_0\sin\mu_0 - 2\sin\vartheta_0\sin^2\mu_0] + \cdots$$

Substitute the latter expression in equation for D_2'' and expand in terms of v/c. This gives:

$$D_2'' = S''\sin\vartheta[\frac{1}{\sin\vartheta_0} - \frac{v}{c}\frac{1}{\sin^2\vartheta_0}\{\sin\vartheta_0\cos\varphi_0 + 2\cos\vartheta_0\sin\mu_0\}$$
$$+\frac{v^2}{c^2}\frac{1}{\sin^3\vartheta_0}\{\sin^2\vartheta_0\cos^2\varphi_0 + 4\cos^2\vartheta_0\sin^2\mu_0$$
$$+2\sin\vartheta_0\cos\vartheta_0\cos\varphi_0\sin\mu_0 + \tfrac{1}{2}\sin^2\vartheta_0\sin^2(\vartheta_0+\varphi_0)$$
$$+\tfrac{1}{2}\sin^2\vartheta_0\sin^2\varphi_0 - \tfrac{1}{2}\sin^2\vartheta_0\cos^2\varphi_0$$
$$+2\sin^2\vartheta_0\sin^2\mu_0 - 2\sin\vartheta_0\cos\vartheta_0\cos\varphi_0\sin\mu_0$$
$$-2\sin^2\vartheta_0\sin\varphi_0\sin\mu_0\} + \cdots],$$

which reduces to:

$$D_2'' = S''\sin\vartheta[\frac{1}{\sin\vartheta_0} - \frac{v}{c}\frac{1}{\sin^2\vartheta_0}\{\sin\vartheta_0\cos\varphi_0 + 2\cos\vartheta_0\sin\mu_0\}$$
$$+\frac{v^2}{c^2}\frac{1}{\sin^3\vartheta_0}\{\tfrac{1}{2}\sin^2\vartheta_0[1+\sin^2(\vartheta_0+\varphi_0)]$$
$$+2\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\sin\mu_0$$
$$+2(1+\cos^2\vartheta_0)\sin^2\mu_0\} + \cdots].$$

Replacing sin Θ and S" by equations 9 and 20 respectively, we obtain as the final expression for D_2'' :

$$D_2'' = S_0''[1 - \frac{v}{c}\frac{1}{\sin\vartheta_0}\{\sin\vartheta_0\cos\varphi_0 + 2\cos\vartheta_0\sin\mu_0\}$$
$$+\frac{v^2}{c^2}\frac{1}{\sin^2\vartheta_0}\{\tfrac{1}{2}\sin^2\vartheta_0 + 2\cos(\vartheta_0+\varphi_0)\sin\mu_0\sin\vartheta_0$$
$$+2(1+\cos^2\vartheta_0)\sin^2\mu_0\} + \cdots].$$

Eqn. (64)

It will be observed that if μ_0 were set equal to zero equation 64 would reduce to equation 63 with the single exception of the different sign in the v/c terms. Also, there is a close resemblance of equations 61 and 62 with equations 63 and 64.

WAVE NUMBERS OF THE RAYS ALONG THE VARIOUS PATHS

Equation 23, derived earlier in this paper, gives the relationship between the reciprocal of the wave length of the light reflected from a moving mirror and the reciprocal of the wave length of the incident light. In order that we may find the reciprocal wave lengths or wave numbers as measured by the moving observer, we must apply the Lorentz transformations to equation 23.

WAVE NUMBER OF THE LIGHT ALONG PATH D_1'

To find the wave number of the light along path D_1' apply equation 23. Let $1/\lambda$ and $1/\lambda_1'$ be the wave numbers of the incident and reflected light, respectively. Referring to equation 27, we see that $v/c \sin\phi$ must be replaced by $-v/c \sin(\theta+\phi)$. Therefore:

$$\frac{1}{\lambda_1'} = \frac{1}{\lambda}\left[1 - 2\frac{v}{c}\sin(\vartheta+\varphi)\cos i_1 + 2\frac{v^2}{c^2}\sin^2(\vartheta+\varphi) + \cdots\right].$$

Replace $\cos i_1$ by its value found on page 38. This gives:

$$\frac{1}{\lambda_1'} = \frac{1}{\lambda}\left[1 - 2\frac{v}{c}\sin(\vartheta+\varphi)\sin\vartheta + 2\frac{v^2}{c^2}\sin(\vartheta+\varphi)\{\sin(\vartheta+\varphi) - \cos\vartheta\sin\varphi\} + \cdots\right].$$

Simplify the second order term and apply equations 3, 5 and 9 .

$$\frac{1}{\lambda_1'} = \frac{1}{\lambda}\left[1 - 2\frac{v}{c}\sin(\vartheta_0+\varphi_0)\sin\vartheta_0 + 2\frac{v^2}{c^2}\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\varphi_0 + \cdots\right].$$

Eqn. (65)

The wave length measured by the moving observer is obtained by applying the Lorentz transformation to λ and λ'_1. From figure I we see that the ray of light which is incident on the plate M makes an angle $(\phi - \alpha)$ with the direction of the velocity. Therefore, applying equation 1, we get:

$$\lambda = \lambda_0 \left[1 - \frac{v^2}{c^2}\cos^2(\varphi - \alpha)\right]^{\frac{1}{2}}.$$

where λ_0 is the wave length of the incident light as measured by the moving observer. Expand $\cos^2(\phi - \alpha)$ and substitute equations 3, 4, 24 and 25. Since $\cos^2(\phi - \alpha)$ is multiplyied by $(v/c)^2$, it is equal to $\cos^2\phi_0$ to the order of approximation that it is needed. Therefore, taking the reciprocal of the last equation and expanding, we obtain:

$$\frac{1}{\lambda} = \frac{1}{\lambda_0}\left[1 + \frac{1}{2}\frac{v^2}{c^2}\cos^2\varphi_0 + \cdots\right]. \qquad \text{Eqn. (66)}$$

In equation 65 replace $1/\lambda$ by the latter expression, this yields:

$$\frac{1}{\lambda'_1} = \frac{1}{\lambda_0}\Big[1 - 2\frac{v}{c}\sin(\vartheta_0+\varphi_0)\sin\vartheta_0 + \frac{v^2}{c^2}\left\{\tfrac{1}{2}\cos^2\varphi_0 + 2\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\varphi_0\right\} + \cdots\Big].$$

Eqn. (67)

It is now necessary to apply the Lorentz transformation to $1/\lambda'_1$ itself. Examining figure I we see that the angle between D'_1 and the direction of the velocity is $(\Theta + \Theta' + \phi - \psi)$. Replacing ψ by its equal $(i'_1 + \Theta' - \frac{\pi}{2})$, we find this angle to be $(\frac{\pi}{2} + \Theta + \phi - i'_1)$. The cosine of this latter angle is found to be $\cos(2\Theta_0 + \phi_0)$ to the order of approximation to which it is needed. Thus:

$$\frac{1}{\lambda'_1} = \frac{1}{\lambda'_{1_0}}\left[1 - \frac{v^2}{c^2}\cos^2(2\vartheta_0+\varphi_0)\right]^{-\frac{1}{2}}$$

Expanding, we obtain:

$$\frac{1}{\lambda_1'} = \frac{1}{\lambda_{1_0}'}\left[1+\frac{1}{2}\frac{v^2}{c^2}\cos^2(2\vartheta_0+\varphi_0)+\cdots\right] \qquad \text{Eqn. (68)}$$

Substituting this latter expression for $1/\lambda_1'$ in equation 67 we obtain as the final equation of the wave number of the light along path D_1':

$$\frac{1}{\lambda_{1_0}'} = \frac{1}{\lambda_0}\Big[1-2\frac{v}{c}\sin(\vartheta_0+\varphi_0)\sin\vartheta_0 + \frac{v^2}{c^2}\Big\{\frac{1}{2}\cos^2\varphi_0-\frac{1}{2}\cos^2(2\vartheta_0+\varphi_0)+2\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\varphi_0\Big\}+\cdots\Big] \qquad \text{Eqn. (69)}$$

WAVE NUMBER OF THE LIGHT ALONG PATH D_2' AND D_3'

Let $1/\lambda_2'$ be the wave number of the light along path D_2'. This will also be the wave number of the light along the path HT or D_3'. The path D_3' will be discussed in detail in a later section. Applying equation 23 and making the same substitution for $v/c \sin\phi$ as was made in equation 33 we obtain:

$$\frac{1}{\lambda_2'} = \frac{1}{\lambda_1'}\left[1-2\frac{v}{c}\cos(2\vartheta_0+\varphi_0)\cos i_2+2\frac{v^2}{c^2}\cos^2(2\vartheta_0+\varphi_0)+\cdots\right]$$

Substitute eauation 37 for $\cos i_2$. This yields:

$$\frac{1}{\lambda_2'} = \frac{1}{\lambda_1'}\left[1-2\frac{v}{c}\cos(2\vartheta_0+\varphi_0)+2\frac{v^2}{c^2}\cos^2(2\vartheta_0+\varphi_0)+\cdots\right]$$

Replace $1/\lambda_1'$ by equation 68.

$$\frac{1}{\lambda_2'} = \frac{1}{\lambda_{1_0}'}\left[1-2\frac{v}{c}\cos(2\vartheta_0+\varphi_0)+\frac{5}{2}\frac{v^2}{c^2}\cos^2(2\vartheta_0+\varphi_0)+\cdots\right]$$

Now, substitute equation 69 for $1/\lambda_{1_0}'$. This yields:

$$\frac{1}{\lambda_2'} = \frac{1}{\lambda_0}\Big[1-2\frac{v}{c}\{\sin(\vartheta_0+\varphi_0)\sin\vartheta_0+\cos(2\vartheta_0+\varphi_0)\} + \frac{v^2}{c^2}\Big\{\frac{1}{2}\cos^2\varphi_0-\frac{1}{2}\cos^2(2\vartheta_0+\varphi_0)+2\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\varphi_0 + 4\cos(2\vartheta_0+\varphi_0)\sin(\vartheta_0+\varphi_0)\sin\vartheta_0+\frac{5}{2}\cos^2(2\vartheta_0+\varphi_0)\Big\}+\cdots\Big]$$

The latter reduces to :

$$\frac{1}{\lambda_2'} = \frac{1}{\lambda_0}\left[1 - 2\frac{v}{c}\cos(\vartheta_0+\varphi_0)\cos\vartheta_0 + \frac{v^2}{c^2}\cos\varphi_0\left\{\frac{1}{2}\cos\varphi_0 + 2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0\right\} + \cdots\right]$$

Eqn. (70)

Finally, we must apply the Lorentz transformation to $1/\lambda_2'$ itself. In figure I, we see that the angle between D_2' or D_3' and the direction of the velocity $(\theta + \theta' + \phi + i_2' + \sigma)$. To the order of approximation that it is needed, the cosine of this latter angle is $\cos(2\theta_0 + \phi_0)$. Therefore, applying equation 1, we obtain

$$\frac{1}{\lambda_2'} = \frac{1}{\lambda_{2_0}'}\left[1 - \frac{v^2}{c^2}\cos^2(2\vartheta_0+\varphi_0)\right]^{-\frac{1}{2}}$$

Expanding, we get:

$$\frac{1}{\lambda_2'} = \frac{1}{\lambda_{2_0}'}\left[1 + \frac{1}{2}\frac{v^2}{c^2}\cos^2(2\vartheta_0+\varphi_0) + \cdots\right]$$

Substituting the latter expression in equation 70 we obtain as the final equation of the wave number along path D_2' and D_3' :

$$\frac{1}{\lambda_{2_0}'} = \frac{1}{\lambda_0}\left[1 - 2\frac{v}{c}\cos(\vartheta_0+\varphi_0)\cos\vartheta_0 + \frac{v^2}{c^2}\left\{\frac{1}{2}\cos^2\varphi_0 - \frac{1}{2}\cos^2(2\vartheta_0+\varphi_0) + 2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0\cos\varphi_0\right\} + \cdots\right]$$

Eqn. (71)

For later reference, we shall write this equation symbolically as:

$$\frac{1}{\lambda_{2_0}'} = \frac{1}{\lambda_0}\left[1 + \frac{v}{c}C + \frac{v^2}{c^2}D + \cdots\right]$$

Eqn. (72)

where C and D represent the coefficients of the $(v/c)^1$ and the $(v/c)^2$ terms, respectively.

WAVE NUMBER OF LIGHT ALONG PATH D_1''

Since the light which is incident upon the mirror M" has not undergone any reflection, its wave number is simply that of the light which the moving observer starts with, namely:

$$\frac{1}{\lambda_0} .$$

WAVE NUMBER OF LIGHT ALONG PATH D_2''

To find the wave number of the light along path D_2'', apply equation 23, Let $1/\lambda_2''$ be the wave number of the light reflected at M" and $1/\lambda_0$ the wave number of the incident light. Referring to equation 47, we see that $v/c \sin \phi$ must be replaced by $-v/c \cos(\phi - \omega)$. Therefore:

$$\frac{1}{\lambda_2''} = \frac{1}{\lambda}\left[1 - 2\frac{v}{c}\cos(\varphi-\omega)\cos j_1 + 2\frac{v^2}{c^2}\cos^2(\varphi-\omega) + \cdots\right]$$

Substitute for $\cos(\phi - \omega)$ and $\cos j_1$, their values given on page 47 . Also replace $1/\lambda$ by its value given in equation 66.

$$\frac{1}{\lambda_2''} = \frac{1}{\lambda_0}\left[1 - 2\frac{v}{c}\cos\varphi_0 + \frac{v^2}{c^2}\left\{\frac{5}{2}\cos^2\varphi_0 - 2\sin\varphi_0 \sin\mu_0\right\} + \cdots\right]$$

Eqn. (73)

Let us now apply the Lorentz transformation to $1/\lambda_2''$. From figure I we found that the angle between D_2'' and the direction of the velocity is $(\phi + j_1' - \omega)$. To the order of approximation that it is needed, the cosine of this latter angle is found to be $\cos \phi_0$. Therefore:

$$\frac{1}{\lambda_2''} = \frac{1}{\lambda_{2_0}''}\left[1 - \frac{v^2}{c^2}\cos^2\varphi_0\right]^{-\frac{1}{2}}$$

Expanding, we obtain:

$$\frac{1}{\lambda_2''} = \frac{1}{\lambda_{2_0}''}\left[1+\frac{1}{2}\frac{v^2}{c^2}\cos^2\varphi_0+\cdots\right].$$

Replacing $1/\lambda_2''$ in equation 73 by this latter expression, we obtain as the final equation of the wave number of the light along path D_2'' :

$$\frac{1}{\lambda_{2_0}''} = \frac{1}{\lambda_0}\left[1-2\frac{v}{c}\cos\varphi_0+2\frac{v^2}{c^2}\{\cos^2\varphi_0-\sin\varphi_0\sin\mu_0\}+\cdots\right]. \qquad \text{Eqn. (74)}$$

WAVE NUMBER OF LIGHT ALONG PATH D_3''

The path D_3'' will be discussed in detail in a later section. Let $1/\lambda_3''$ be the wave number of the light along the path D_3'' . Applying equation 23, and making the same substitution for $v/c \sin \phi$ as was made in equation 53, we obtain:

$$\frac{1}{\lambda_3''} = \frac{1}{\lambda_2''}\left[1+2\frac{v}{c}\sin(\vartheta+\varphi)\cos j_2+2\frac{v^2}{c^2}\sin^2(\vartheta+\varphi)+\cdots\right].$$

Replace $\cos j_2$ by its value given on page 51. Also substitute equation 5 for $\sin(\Theta+\phi)$. This yields:

$$\frac{1}{\lambda_3''} = \frac{1}{\lambda_2''}\Big[1+2\frac{v}{c}\sin(\vartheta_0+\varphi_0)\sin\vartheta_0$$
$$+2\frac{v^2}{c^2}\sin(\vartheta_0+\varphi_0)\{\sin(\vartheta_0+\varphi_0)-\cos\vartheta_0\sin\varphi_0+2\cos\vartheta_0\sin\mu_0\}+\cdots\Big],$$

Simplifying, this becomes:

$$\frac{1}{\lambda_3''} = \frac{1}{\lambda_2''}\Big[1+2\frac{v}{c}\sin(\vartheta_0+\varphi_0)\sin\vartheta_0$$
$$+2\frac{v^2}{c^2}\sin(\vartheta_0+\varphi_0)\{\sin\vartheta_0\cos\varphi_0+2\cos\vartheta_0\sin\mu_0\}+\cdots\Big]. \qquad \text{Eqn. (75)}$$

Replacing $1/\lambda_2''$ by its value given in equation 73, yields:

$$\frac{1}{\lambda_3''} = \frac{1}{\lambda_0}\Big[1+2\frac{v}{c}\{\sin(\vartheta_0+\varphi_0)\sin\vartheta_0-\cos\varphi_0\} + \frac{v^2}{c^2}\{\tfrac{5}{2}\cos^2\varphi_0 - 2\sin\varphi_0\sin\mu_0 - 4\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\varphi_0 + 2\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\varphi_0 + 4\sin(\vartheta_0+\varphi_0)\cos\vartheta_0\sin\mu_0\}+\cdots\Big].$$

The latter reduces to :

$$\frac{1}{\lambda_3''} = \frac{1}{\lambda_0}\Big[1-2\frac{v}{c}\cos(\vartheta_0+\varphi_0)\cos\vartheta_0 + \frac{v^2}{c^2}\{\tfrac{1}{2}\cos^2\varphi_0 + 2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0\cos\varphi_0 + 2\sin(2\vartheta_0+\varphi_0)\sin\mu_0\}+\cdots\Big]$$

Eqn. (76)

Finally, we must apply the Lorentz transformation to $1/\lambda_3''$. Referring to figure I we see that the angle between D_3'' and the direction of the velocity is $(j_2' + \frac{\pi}{2} - \theta - \phi)$. To the order of approximation that it is needed, the cosine of this latter angle is found to be $\cos(2\theta_0 + \phi_0)$. Therefore, substituting in equation 1, we get:

$$\frac{1}{\lambda_3''} = \frac{1}{\lambda_{3_0}''}\Big[1-\frac{v^2}{c^2}\cos^2(2\vartheta_0+\varphi_0)\Big]^{-\frac{1}{2}}.$$

which becomes upon expansion:

$$\frac{1}{\lambda_3''} = \frac{1}{\lambda_{3_0}''}\Big[1+\frac{1}{2}\frac{v^2}{c^2}\cos^2(2\vartheta_0+\varphi_0)+\cdots\Big].$$

Replace $1/\lambda_3''$ in equation 76 by this latter expression.

$$\frac{1}{\lambda_{3_0}''} = \frac{1}{\lambda_0}\Big[1-2\frac{v}{c}\cos(\vartheta_0+\varphi_0)\cos\vartheta_0 + \frac{v^2}{c^2}\{\tfrac{1}{2}\cos^2\varphi_0 - \tfrac{1}{2}\cos^2(2\vartheta_0+\varphi_0) + 2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0\cos\varphi_0 + 2\sin(2\vartheta_0+\varphi_0)\sin\mu_0\}+\cdots\Big].$$

Eqn. (77)

For later référence, we shall write this equation symbolically as:

$$\frac{1}{\lambda_{3_0}''} = \frac{1}{\lambda_0}\Big[1+\frac{v}{c}A+\frac{v^2}{c^2}B+\cdots\Big].$$

Eqn. (78)

where A and B represent the coefficient of the $(v/c)^1$ and the $(v/c)^2$ terms respectively.

Comparing equations 71 and 77 we see that except for the term involving μ_0 the two final wave numbers are equal to the second order of v/c.

NUMBER OF WAVES IN THE PATHS D_1', D_2', D_1'' AND D_2''

In order to compare the number of waves in paths AC + CH − HT and AP + PL − LT it is necessary to multiply each part of the path by the wave number of the light contained within that path.

Let N_1' be defined as the number of wave lengths in the path D_1'.

Then:

$$N_1' = \frac{D_1'}{\lambda_{1_0}'}$$

To find N_1' multiply equation 61 by equation 69.

$$\begin{aligned} N_1' = \frac{S_0'}{\lambda_0}\Big[1 &+ \frac{v}{c}\{\cos(2\vartheta_0+\varphi_0) - 2\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\} \\ &+ \frac{v^2}{c^2}\{\tfrac{1}{2} - 2\cos(2\vartheta_0+\varphi_0)\sin(\vartheta_0+\varphi_0)\sin\vartheta_0 + \tfrac{1}{2}\cos^2\varphi_0 \\ &\quad - \tfrac{1}{2}\cos^2(2\vartheta_0+\varphi_0) + 2(\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\varphi_0\}\cdots\Big]. \end{aligned}$$

Upon simplification, this becomes:

$$\begin{aligned} N_1' = \frac{S_0'}{\lambda_0}\Big[1 &+ \frac{v}{c}\{\cos(2\vartheta_0+\varphi_0) - 2\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\} \\ &+ \frac{v^2}{c^2}\{\tfrac{1}{2}\sin^2(2\vartheta_0+\varphi_0) + \tfrac{1}{2}\cos^2\varphi_0 \\ &\quad + 2\sin(\vartheta_0+\varphi_0)\sin\vartheta_0[\cos\varphi_0 - \cos(2\vartheta_0+\varphi_0)]\} + \cdots\Big]. \end{aligned}$$

Eqn. (79)

Let N_2' be defined as the number of wave lengths in the path D_2'

Then:

$$N_2' = \frac{D_2'}{\lambda_{2_0}'}$$

To find N_2' multiply equation 62 by equation 71.

$$N_2' = \frac{S_0'}{\lambda_0}[1 - \frac{v}{c}\{\cos(2\vartheta_0+\varphi_0) + 2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0\}$$
$$+\frac{v^2}{c^2}\{\frac{1}{2} + 2\cos(2\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\cos\vartheta_0 + \frac{1}{2}\cos^2\varphi_0$$
$$+2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0\cos\varphi_0 - \frac{1}{2}\cos^2(2\vartheta_0+\varphi_0)\}+\cdots]$$

This reduces to :

$$N_2' = \frac{S_0'}{\lambda_0}[1 - \frac{v}{c}\{\cos(2\vartheta_0+\varphi_0) + 2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0$$
$$+\frac{v^2}{c^2}\{\frac{1}{2}\sin^2(2\vartheta_0+\varphi_0) + \frac{1}{2}\cos^2\varphi_0$$
$$+2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0[\cos\varphi_0 + \cos(2\vartheta_0+\varphi_0)]\}+\cdots]$$

Eqn. (80)

Let N_1'' be defined as the number of wave lengths in path D_1''.

Then:

$$N_1'' = \frac{D_1''}{\lambda_0}$$

To find N_1'' multiply equation 63 by $1/\lambda_0$

$$N_1'' = \frac{S_0''}{\lambda_0}[1 + \frac{v}{c}\cos\varphi_0 + \frac{1}{2}\frac{v^2}{c^2}+\cdots].$$

Eqn. (81)

Let N_2'' be defined as the number of wave lengths in path D_2''.

Then:

$$N_2'' = \frac{D_2''}{\lambda_{2_0}''}$$

To find N_2'' multiply equation 64 by equation 74.

$$N_2'' = \frac{S_0''}{\lambda_0}[1 - \frac{v}{c}\frac{1}{\sin\vartheta_0}\{\sin\vartheta_0\cos\varphi_0 + 2\cos\vartheta_0\sin\mu_0 + 2\sin\vartheta_0\cos\varphi_0\}$$
$$+\frac{v^2}{c^2}\frac{1}{\sin^2\vartheta_0}\{\frac{1}{2}\sin^2\vartheta_0 + 2\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\sin\mu_0$$
$$+2(1+\cos^2\vartheta_0)\sin^2\mu_0 + 2\sin^2\vartheta_0\cos^2\varphi_0$$
$$+4\sin\vartheta_0\cos\vartheta_0\cos\varphi_0\sin\mu_0 + 2\sin^2\vartheta_0\cos^2\varphi_0$$
$$-2\sin^2\vartheta_0\sin\varphi_0\sin\mu_0\}+\cdots]$$

Upon simplification, this reduces to :

$$N_2'' = \frac{S_0''}{\lambda_0}\Big[1 - \frac{v}{c}\frac{1}{\sin\vartheta_0}\{3\sin\vartheta_0\cos\varphi_0 + 2\cos\vartheta_0\sin\mu_0\}$$
$$+\frac{v^2}{c^2}\frac{1}{\sin^2\vartheta_0}\{\tfrac{1}{2}\sin^2\vartheta_0(1+8\cos^2\varphi_0) + 2\sin\vartheta_0\sin\mu_0(2\cos(\vartheta_0+\varphi_0)+\cos\vartheta_0\cos\varphi_0)$$
$$+2(1+\cos^2\vartheta_0)\sin^2\mu_0\} + \cdots\Big]$$ Eqn. (82)

Heving obtained the number of wave lengths along the paths D_1', D_2' D_1'' and D_2''; in terms of the wave length of the light with which the moving observer is working, we may now find the difference between the number of wave lenghs in corresponding parts of the two paths. Let:

$$N_1'' - N_1' = N_1$$

Substracting equation 79 from equation 81, and multiplying both sides by λ_0, we get:

$$N_1\lambda_0 = (S_0''-S_0') + \frac{v}{c}\big[S_0''\cos\varphi_0 - S_0'\{\cos(2\vartheta_0+\varphi_0) - 2\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\}\big]$$
$$+\frac{v^2}{c^2}\Big[\tfrac{1}{2}S_0'' - S_0'\{\tfrac{1}{2}\sin^2(2\vartheta_0+\varphi_0) + \tfrac{1}{2}\cos^2\varphi_0$$
$$+2\sin(\vartheta_0+\varphi_0)\sin\vartheta_0[\cos\varphi_0 - \cos(2\vartheta_0+\varphi_0)]\}\Big] + \cdots$$

Eqn. (83)

Similarly, let :

$$N_2'' - N_2' = N_2$$

Substracting equation 80 from equation 82, and multiplying both sides by λ_0 we get:

$$N_2\lambda_0 = (S_0''-S_0') - \frac{v}{c}\Big[S_0''\frac{1}{\sin\vartheta_0}\{3\sin\vartheta_0\cos\varphi_0 + 2\cos\vartheta_0\sin\mu_0\}$$
$$-S_0'\{\cos(2\vartheta_0+\varphi_0) + 2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0\}\Big]$$
$$+\frac{v^2}{c^2}\Big[S_0''\frac{1}{\sin^2\vartheta_0}\{\tfrac{1}{2}\sin^2\vartheta_0(1+8\cos^2\varphi_0) + 2\sin\vartheta_0\sin\mu_0[2\cos(\vartheta_0+\varphi_0)+\cos\vartheta_0\cos\varphi_0]$$
$$+2(1+\cos^2\vartheta_0)\sin^2\mu_0\}$$
$$-S_0'\{\tfrac{1}{2}\sin^2(2\vartheta_0+\varphi_0) + \tfrac{1}{2}\cos^2\varphi_0$$
$$+2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0[\cos\varphi_0 + \cos(2\vartheta_0+\varphi_0)]\}\Big] + \cdots$$

Eqn. (84)

Let us now add equations 83 and 84 to get the total difference in path length along the first two parts of each path.

$$
\begin{aligned}
(N_1+N_2)\lambda_0 = 2(S_0''-S_0') &+ \frac{v}{c}\Big[-S_0''\frac{2}{\sin\vartheta_0}\{\sin\vartheta_0\cos\varphi_0+\cos\vartheta_0\sin\mu_0\}-2S_0'\cos\varphi_0\Big] \\
&+\frac{v^2}{c^2}\Big[S_0''\frac{1}{\sin^2\vartheta_0}\{\sin^2\vartheta_0(1+4\cos^2\varphi_0) \\
&\qquad +2\sin\vartheta_0\sin\mu_0[2\cos(\vartheta_0+\varphi_0)+\cos\vartheta_0\cos\varphi_0] \\
&\qquad +2(1+\cos^2\vartheta_0)\sin^2\mu_0\} \\
&\qquad -S_0'\{\sin^2(2\vartheta_0+\varphi_0)+3\cos^2\varphi_0+2\cos^2(2\vartheta_0+\varphi_0)\}\Big]+\cdots
\end{aligned}
$$

Simplifying, we obtain:

$$
\begin{aligned}
(N_1+N_2)\lambda_0 = 2(S_0''-S_0') &+\frac{v}{c}\Big[-2(S_0''-S_0')\cos\varphi_0-2S_0''\frac{1}{\sin\vartheta_0}\cos\vartheta_0\sin\mu_0\Big] \\
&+\frac{v^2}{c^2}\Big[(S_0''-S_0')(1+3\cos^2\varphi_0) \\
&\qquad +S_0''\frac{2}{\sin^2\vartheta_0}\{\sin\vartheta_0\sin\mu_0[2\cos(\vartheta_0+\varphi_0)+\cos\vartheta_0\cos\varphi_0] \\
&\qquad +\sin^2\vartheta_0\cos^2\varphi_0+(1+\cos^2\vartheta_0)\sin^2\mu_0\} \\
&\qquad -S_0'\cos^2(2\vartheta_0+\varphi_0)\Big]+\cdots
\end{aligned}
\qquad \text{Eqn. (85)}
$$

DIFFERENCE BETWEEN NUMBER OF WAVE LENGTHS IN THE TWO FINAL PATHS

In examining the equation for $\sin\epsilon$ (Eqn.58), we find that to the first power of v/c this angle is a function of μ_0, the angle introduced at the mirror M". The second order term of $\sin\epsilon$, we find, is due partially to μ_0 and partially to the angles θ_0 and ϕ_0. Because of this small angle ϵ the two final rays seem to come from a point T in figure I. Hence, in order to evaluate the number of wave lengths in the path designated by the single primes, it is necessary to substract from N_2' the number of wave lengths contained within the distance HT. Let:

$$HT = D_3'$$

Similarly, to evaluate the number of wave length in the path designated by the double primes, it is necessary to substract from N''_2 the number of wave lengths contained within the distance LT. Let:

$$LT = D''_3 .$$

We shall now evaluate the distances D'_3 and D''_3. In triangle TLG of figure I, we have by the sine law:

$$\frac{\sin \epsilon}{LG} = \frac{\sin(\frac{\pi}{2} - i_3)}{D''_3} = \frac{\sin(\frac{\pi}{2} + j'_2)}{D'_3 + HG} .$$

Eqn. (86)

Solving for D''_3, we get:

$$D''_3 = LG \frac{\cos i_3}{\sin \epsilon}$$

Replacing i_3 by its value $(\frac{\pi}{2} - \theta' - \sigma - i'_2)$, we get:

$$D''_3 = LG \frac{\sin(\vartheta' + \sigma + i'_2)}{\sin \epsilon}$$

Substitute for $\sin \epsilon$ its equal $v/c \sin \epsilon'$. This yields:

$$D''_3 = \frac{LG}{\frac{v}{c}} \frac{\sin(\vartheta' + \sigma + i'_2)}{\sin \epsilon'}$$

The wave number of the light along the path D''_3 was represented in equation 78 as :

$$\frac{1}{\lambda''_{3_0}} = \frac{1}{\lambda_0} \left(1 + \frac{v}{c} A + \frac{v^2}{c^2} B + \cdots\right)$$

Let N''_3 be defined as the number of waves in the path D''_3, Then:

$$N_3'' = \frac{D_3''}{\lambda''_{3_0}}$$

Therefore:

$$N_3'' = \frac{LG}{\frac{v}{c}\lambda_0}\left(1+\frac{v}{c}A+\frac{v^2}{c^2}B+\cdots\right)\frac{\sin(\theta'+\sigma+i_2')}{\sin\epsilon'} .$$

Now, substitute equations 44 and 60 for $\sin(\theta'+\sigma+i_2')$ and $1/\sin\epsilon'$.

$$N_3'' = \frac{LG}{\frac{v}{c}\lambda_0}\left(1+\frac{v}{c}A+\frac{v^2}{c^2}B\right)\left(J+\frac{v}{c}K+\frac{v^2}{c^2}M\right)\left(E+\frac{v}{c}F+\frac{v^2}{c^2}P\right).$$

Multiplying, we obtain:

$$N_3'' = \frac{LG}{\frac{v}{c}\lambda_0}\Big[EJ+\frac{v}{c}(EK+FJ+EAJ) + \frac{v^2}{c^2}(EM+KF+JP+EAK+BJE+FAJ)+\cdots\Big].$$

Eqn. (87)

Similarly, from equation 86, we obtain:

$$D_3' = LG\frac{\cos j_2'}{\sin\epsilon} - HG .$$

Substitute for $\sin\epsilon$ its value $v/c \sin\epsilon'$ Then:

$$D_3' = \frac{LG}{\frac{v}{c}}\,\frac{\cos j_2'}{\sin\epsilon'} - HG .$$

The wave number of the light along the path D_3' was represented in equation 72, as:

$$\frac{1}{\lambda_{2_0}'} = \frac{1}{\lambda_0}\left[1+\frac{v}{c}C+\frac{v^2}{c^2}D+\cdots\right].$$

Let N_3' be defined as the number of waves in the path D_3'. Then:

$$N_3' = \frac{D_3'}{\lambda_{2_0}'}$$

Therefore:

$$N_3' = \frac{LG}{\frac{v}{c}\lambda_0}\left(1+\frac{v}{c}C+\frac{v^2}{c^2}D+\cdots\right)\frac{\cos j_2'}{\sin\epsilon'} - \frac{HG}{\lambda_0}\left(1+\frac{v}{c}C+\frac{v^2}{c^2}D+\cdots\right).$$

Substitute equations 57 and 58 for $\cos j_2'$ and $\sin\epsilon'$.

$$N_3' = \frac{LG}{\frac{v}{c}\lambda_0}\left(1+\frac{v}{c}C+\frac{v^2}{c^2}D\right)\left(J+\frac{v}{c}K+\frac{v^2}{c^2}M\right)\left(R+\frac{v}{c}S+\frac{v^2}{c^2}T\right)$$

$$-\frac{HG}{\lambda_0}\left(1+\frac{v}{c}C+\frac{v^2}{c^2}D\right).$$

Multiplying, we obtain:

$$N_3' = \frac{LG}{\frac{v}{c}\lambda_0}\Big[JR+\frac{v}{c}(RK+JS+RJC)$$

$$+\frac{v^2}{c^2}(RM+SK+JT+RCK+RJD+SJC)+\cdots\Big]$$

$$-\frac{HG}{\lambda_0}\left(1+\frac{v}{c}C+\frac{v^2}{c^2}D+\cdots\right). \qquad \text{Eqn. (88)}$$

Let us now subtract the number of wave lenths in the path D_3' from the number in the path D_3''. Define N_3, as:

$$N_3 = N_3'' - N_3'$$

Then:

$$N_3 = \frac{LG}{\frac{v}{c}\lambda_0}\Big[J(E-R)+\frac{v}{c}\{K(E-R)+J(F-S)+J(EA-RC)\}$$

$$+\frac{v^2}{c^2}\{M(E-R)+K(F-S)+J(P-T)+K(EA-RC)$$

$$+J(BE-RD)+J(FA-SC)\}\Big]+\frac{HG}{\lambda_0}\left(1+\frac{v}{c}C+\frac{v^2}{c^2}D\right)+\cdots \qquad \text{Eqn. (89)}$$

It will be convenient at this point to summarize the values of the symbols used in the above equation.

$$A = -2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0$$

$$B = \frac{1}{2}\cos^2\varphi_0 - \frac{1}{2}\cos^2(2\vartheta_0+\varphi_0) + 2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0\cos\varphi_0$$

$$+2\sin(2\vartheta_0+\varphi_0)\sin\mu_0$$

$C = -2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0$

$D = \frac{1}{2}\cos^2\varphi_0 - \frac{1}{2}\cos^2(2\vartheta_0+\varphi_0) + 2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0\cos\varphi_0$

$E = \sin\vartheta_0$

$F = \sin(2\vartheta_0+\varphi_0)\cos\vartheta_0$

$$P = -\frac{1}{2}\sin^2(2\vartheta_0+\varphi_0)\sin\vartheta_0 + \frac{1}{2}\cos^2(\vartheta_0+\varphi_0)\sin\vartheta_0 - \frac{1}{2}\sin^2\varphi_0\sin\vartheta_0$$
$$-2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin^2\vartheta_0\cos\vartheta_0 + 4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\cos\vartheta_0$$

$$J = \frac{1}{2\sin\mu_0}$$

$$K = -\frac{1}{4\sin^2\mu_0}\Big[\frac{1}{2}\sin\vartheta_0\cos\vartheta_0\{\sin^2(2\vartheta_0+\varphi_0) - \sin^2(\vartheta_0+\varphi_0)\}$$
$$-4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0) + 2\cos(2\vartheta_0+\varphi_0)\sin\mu_0\Big]$$

M will drop out of the equation, therefore, it has not been evaluated.

$R = \sin\vartheta_0$

$S = \cos\vartheta_0[\sin(2\vartheta_0+\varphi_0) + 2\sin\mu_0]$

$$T = -\frac{1}{2}\sin^2(\vartheta_0+\varphi_0)\sin\vartheta_0 + \frac{1}{2}\sin\vartheta_0\cos^2\varphi_0 - \frac{1}{2}\sin\vartheta_0\sin^2\varphi_0$$
$$+4\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\sin\varphi_0 - 4\sin^2(\vartheta_0+\varphi_0)\sin\vartheta_0\cos^2\vartheta_0$$
$$+2\sin\vartheta_0\sin\varphi_0\sin\mu_0 + 2\cos\vartheta_0\cos\varphi_0\sin\mu_0$$
$$-8\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\sin\mu_0 - 2\sin\vartheta_0\sin^2\mu_0$$

Let us now evaluate the various terms in equation 89.

$$(E-R)=0 \quad ;(A-C)=0 \quad \text{and} \quad (EA-RC)=0 .$$

$$J(F-S) = -\cos\vartheta_0$$

$$K(F-S) = \frac{\cos\vartheta_0}{2\sin\mu_0}\Big[\frac{1}{2}\sin\vartheta_0\cos\vartheta_0\{\sin^2(2\vartheta_0+\varphi_0) - \sin^2(\vartheta_0+\varphi_0)\}$$
$$-4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0) + 2\cos(2\vartheta_0+\varphi_0)\sin\mu_0\Big]$$

$$J(P-T) = \frac{1}{2\sin\mu_0}\Big[-\tfrac{1}{2}\sin^2(2\vartheta_0+\varphi_0)\sin\vartheta_0 + \tfrac{1}{2}\cos^2(\vartheta_0+\varphi_0)\sin\vartheta_0 - \tfrac{1}{2}\sin^2\varphi_0\sin\vartheta_0$$
$$-2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin^2\vartheta_0\cos\vartheta_0 + 4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\cos\vartheta_0$$
$$+\tfrac{1}{2}\sin^2(\vartheta_0+\varphi_0)\sin\vartheta_0 - \tfrac{1}{2}\sin\vartheta_0\cos^2\varphi_0 + \tfrac{1}{2}\sin\vartheta_0\sin^2\varphi_0$$
$$-4\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\sin\varphi_0 + 4\sin^2(\vartheta_0+\varphi_0)\sin\vartheta_0\cos^2\vartheta_0$$
$$-2\sin\vartheta_0\sin\varphi_0\sin\mu_0 - 2\cos\vartheta_0\cos\varphi_0\sin\mu_0$$
$$+8\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\sin\mu_0 + 2\sin\vartheta_0\sin^2\mu_0\Big].$$

The latter expression reduces to:

$$J(P-T) = \frac{1}{2\sin\mu_0}\Big[-\tfrac{1}{2}\sin^2(2\vartheta_0+\varphi_0)\sin\vartheta_0 + \tfrac{1}{2}\sin\vartheta_0\sin^2\varphi_0$$
$$+4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\cos\vartheta_0 + 2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin^2\vartheta_0\cos\vartheta_0$$
$$-2\sin\mu_0\{\sin(2\vartheta_0+\varphi_0)\sin\vartheta_0 - \cos(2\vartheta_0+\varphi_0)\cos\vartheta_0\}$$
$$+2\sin\vartheta_0\sin^2\mu_0\Big].$$

$$J(BE-RD) = JE(B-D)$$
$$= \sin(2\vartheta_0+\varphi_0)\sin\vartheta_0.$$

$$J(FA-SC) = JA(F-S)$$
$$= 2\cos(\vartheta_0+\varphi_0)\cos^2\vartheta_0.$$

Having evaluated all the terms appearing in equation 89, let us substitute them in that equation.

$$N_3\lambda_0 = \frac{LG}{\frac{v}{c}}\Big[-\tfrac{v}{c}\cos\vartheta_0 + \tfrac{v^2}{c^2}\Big(\frac{1}{\sin\mu_0}\Big\{\tfrac{1}{4}\sin\vartheta_0\cos^2\vartheta_0\sin^2(2\vartheta_0+\varphi_0)$$
$$-\tfrac{1}{4}\sin\vartheta_0\cos^2\vartheta_0\sin^2(\vartheta_0+\varphi_0) - 2\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\cos\vartheta_0$$
$$-\tfrac{1}{4}\sin^2(2\vartheta_0+\varphi_0)\sin\vartheta_0 + \tfrac{1}{4}\sin\vartheta_0\sin^2\varphi_0$$
$$+2\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\cos\vartheta_0 + \sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin^2\vartheta_0\cos\vartheta_0\Big\}$$
$$+\{\cos(2\vartheta_0+\varphi_0)\cos\vartheta_0 - \sin(2\vartheta_0+\varphi_0)\sin\vartheta_0 + \cos(2\vartheta_0+\varphi_0)\cos\vartheta_0$$
$$+\sin(2\vartheta_0+\varphi_0)\sin\vartheta_0 + 2\cos(\vartheta_0+\varphi_0)\cos^2\vartheta_0\}$$
$$+\sin\mu_0\sin\vartheta_0\Big)+\cdots\Big] + HG\Big[1+\tfrac{v}{c}C+\tfrac{v^2}{c^2}D+\cdots\Big].$$

Upon simplification, this becomes:

$$N_3\lambda_0 = \frac{LG}{\frac{v}{c}}\Big[-\tfrac{v}{c}\cos\vartheta_0 + \tfrac{v^2}{c^2}\Big(\frac{1}{\sin\mu_0}\Big\{-\tfrac{1}{4}\sin\vartheta_0\cos^2\vartheta_0\sin^2(\vartheta_0+\varphi_0) - \tfrac{1}{4}\sin^2(2\vartheta_0+\varphi_0)\sin^3\vartheta_0$$
$$+\tfrac{1}{4}\sin\vartheta_0\sin^2\varphi_0 + \sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin^2\vartheta_0\cos\vartheta_0\Big\}$$
$$+\{2\cos(2\vartheta_0+\varphi_0)\cos\vartheta_0 + 2\cos(\vartheta_0+\varphi_0)\cos^2\vartheta_0\}$$
$$+\sin\mu_0\sin\vartheta_0\Big)+\cdots\Big] + HG\Big[1+\tfrac{v}{c}C+\tfrac{v^2}{c^2}D+\cdots\Big].$$

Eqn. (90)

Let us now evaluate LG. Referring to figure I we see that:

$$LG = RN + GN - RL. \qquad \text{Eqn. (91)}$$

In the right triangle HNR:

$$RN = HR \sin\left(\tfrac{\pi}{2} - \vartheta - \varphi\right)$$
$$= HR \cos(\vartheta + \varphi).$$

Also, in the right triangle HNG

$$GN = HR \cos\left(\tfrac{\pi}{2} - \vartheta - \varphi\right) \tan i_3 .$$

Substitute for i_3 its value $(\frac{\pi}{2} - \theta' - \sigma - i_2')$. This yields:

$$GN = HR \sin(\vartheta + \varphi)\, \mathrm{ctn}(\vartheta' + \sigma + i_2')$$

Looking at the line AHR, we see that :

$$HR = v(t_1'' + t_2'' - t_1' - t_2')$$
$$= \tfrac{v}{c}(D_1'' + D_2'' - D_1' - D_2').$$

Let us replace HR by this value in the equations for RN and GN.

$$RN = \tfrac{v}{c}(D_1'' + D_2'' - D_1' - D_2') \cos(\vartheta + \varphi)$$

and

$$GN = \tfrac{v}{c}(D_1'' + D_2'' - D_1' - D_2') \sin(\vartheta + \varphi)\, \mathrm{ctn}(\vartheta' + \sigma + i_2').$$

To evaluate RL, we observe that:

$$RL = BL - BR$$

In triangle BLP we have, by the sine law :

$$\frac{\sin(j_1' - \omega)}{BL} = \frac{\sin(\pi - \vartheta)}{ct_2''}$$

or

$$BL = ct_2'' \frac{\sin(j_1' - \omega)}{\sin\vartheta}.$$

Also, in triangle BFR:

$$\frac{\sin\varphi}{BR} = \frac{\sin\vartheta}{vt_2''}$$

or

$$BR = vt_2''\,\frac{\sin\varphi}{\sin\vartheta}\;.$$

Therefore:

$$RL = ct_2''\,\frac{\sin(j_1'-\omega)}{\sin\vartheta} - vt_2''\,\frac{\sin\varphi}{\sin\vartheta}$$

or

$$RL = D_2''\,\frac{[\sin(j_1'-\omega) - \frac{v}{c}\sin\varphi]}{\sin\vartheta}$$

Now, in equation 91 replace RN, GN and RL by their values just obtained.

Also observe that in equation 90 we need $LG/\frac{v}{c}$. Therefore:

$$\frac{LG}{\frac{v}{c}} = (D_1''+D_2''-D_1'-D_2')[\cos(\vartheta+\varphi)+\sin(\vartheta+\varphi)\operatorname{ctn}(\vartheta'+\sigma+i_2')]$$
$$-D_2''\,\frac{[\sin(j_1'-\omega) - \frac{v}{c}\sin\varphi]}{\frac{v}{c}\sin\vartheta}$$

Eqn. (92)

Again, observing equation 90 we see that $LG/\frac{v}{c}$ is needed only up to the first power of v/c , since $(v/c)^1$ is the lowest power of its coefficient.

$$\cos(\vartheta+\varphi) + \sin(\vartheta+\varphi)\operatorname{ctn}(\vartheta'+\sigma+i_2')$$
$$= \frac{\sin(\vartheta+\vartheta'+\varphi+\sigma+i_2')}{\sin(\vartheta'+\sigma+i_2')}$$

Substituting the value for $\sin(\theta+\theta'+\phi+\sigma+i_2')$ given on the top of page 58, we get:

$$\cos(\vartheta+\varphi) + \sin(\vartheta+\varphi)\operatorname{ctn}(\vartheta'+\sigma+i_2') = \frac{\sin(\vartheta+\vartheta'+\varphi)}{\frac{D_2'}{s'}\,\sin(\vartheta'+\sigma+i_2')}$$

From the equation at the bottom of page 59 we see that :

$$\frac{1}{\frac{D_2'}{S'}} = 1 + \frac{v}{c}\cos(2\vartheta_0+\varphi_0)+\cdots .$$

Multiplying this latter expression by equations 7 and 46, we get:

$$\frac{\sin(\vartheta+\vartheta'+\varphi)}{\frac{D_2'}{S'}\sin(\vartheta'+\sigma+i_2')} = \sin(2\vartheta_0+\varphi_0)\Big[\frac{1}{\sin\vartheta_0}$$
$$-\frac{v}{c}\frac{1}{\sin^2\vartheta_0}\{\cos\vartheta_0\sin(2\vartheta_0+\varphi_0)-\cos(2\vartheta_0+\varphi_0)\sin\vartheta_0\}\Big]$$

which reduces to:

$$\frac{\sin(\vartheta+\vartheta'+\varphi)}{\frac{D_2'}{S'}\sin(\vartheta'+\sigma+i_2')} = \sin(2\vartheta_0+\varphi_0)\Big[\frac{1}{\sin\vartheta_0}-\frac{v}{c}\frac{\sin(\vartheta_0+\varphi_0)}{\sin^2\vartheta_0}+\cdots\Big].$$

Eqn. (93)

Subtracting equations 61 and 62 from equations 63 and 64, we obtain:

$$(D_1''+D_2''-D_1'-D_2') = 2(S_0''-S_0')-2\frac{v}{c}S_0''\frac{\cos\vartheta_0}{\sin\vartheta_0}\sin\mu_0+\cdots .$$

Eqn. (94)

Let us now multiply equation 93 by equation 94. This gives:

$$(D_1''+D_2''-D_1'-D_2')[\cos(\vartheta+\varphi)+\sin(\vartheta+\varphi)\operatorname{ctn}(\vartheta'+\sigma+i_2')]$$
$$= 2(S_0''-S_0')\frac{\sin(2\vartheta_0+\varphi_0)}{\sin\vartheta_0}$$
$$-\frac{v}{c}\frac{1}{\sin^2\vartheta_0}\{2(S_0''-S_0')\sin(2\vartheta_0+\varphi_0)\sin(\vartheta_0+\varphi_0)+2S_0''\sin(2\vartheta_0+\varphi_0)\cos\vartheta_0\sin\mu_0\}+\cdots$$

Eqn. (95)

We must now evaluate :

$$D_2''\frac{[\sin(j_1'-\omega)-\frac{v}{c}\sin\varphi]}{\frac{v}{c}\sin\vartheta} .$$

First, we shall treat $[\sin(j_1'-\omega)-v/c\ \sin\phi]$.

$$\sin(j_1' - \omega) = \sin j_1' \cos\omega - \cos j_1' \sin\omega$$

Substitute equations 4, 17,18, 49 and 50.

$$\sin(j_1' - \omega) - \frac{v}{c}\sin\varphi = \frac{v}{c}(\sin\varphi_0 - \sin\mu_0) + \frac{v^2}{c^2}\cos\varphi_0(\sin\varphi_0 - 2\sin\mu_0)$$
$$- \frac{v}{c}\sin\mu_0 + \frac{v^2}{c^2}\sin\varphi_0\cos\varphi_0 - \frac{v}{c}\sin\varphi_0 \quad .$$

Collecting terms, we obtain:

$$\sin(j_1' - \omega) - \frac{v}{c}\sin\varphi = -2\frac{v}{c}\sin\mu_0 + 2\frac{v^2}{c^2}\cos\varphi_0(\sin\varphi_0 - \sin\mu_0) + \cdots .$$

Cancelling a v/c and substituting equation 9 for sinθ and equation 64 for D_2'' yields:

$$D_2'' \frac{[\sin(j_1' - \omega) - \frac{v}{c}\sin\varphi]}{\frac{v}{c}\sin\vartheta} = \frac{S_0''}{\sin\vartheta_0}\left[1 + \frac{\frac{v}{c}}{\sin\vartheta_0}\{\sin\vartheta_0\cos\varphi_0 + 2\cos\vartheta_0\sin\mu_0\}\right]$$
$$\left[-2\sin\mu_0 + 2\frac{v}{c}\cos\varphi_0(\sin\varphi_0 - \sin\mu_0)\right].$$

Multiplying and collecting terms, reduces the latter to:

$$D_2'' \frac{[\sin(j_1' - \omega) - \frac{v}{c}\sin\varphi]}{\frac{v}{c}\sin\vartheta} = \frac{2S_0''}{\sin\vartheta_0}\Big[-\sin\mu_0$$
$$+ \frac{v}{c}\frac{1}{\sin\vartheta_0}\{\sin\vartheta_0\sin\varphi_0\cos\varphi_0 + 2\cos\vartheta_0\sin^2\mu_0\} + \cdots\Big].$$

Substracting this latter expression from equation 95, we obtain:

$$\frac{LG}{\frac{v}{c}} = 2(S_0'' - S_0')\frac{\sin(2\vartheta_0 + \varphi_0)}{\sin\vartheta_0} + 2S_0''\frac{\sin\mu_0}{\sin\vartheta_0}$$
$$+ \frac{v}{c}\frac{1}{\sin^2\vartheta_0}\Big[-2(S_0'' - S_0')\sin(2\vartheta_0 + \varphi_0)\sin(\vartheta_0 + \varphi_0)$$
$$-2S_0''\{\sin(2\vartheta_0 + \varphi_0)\cos\vartheta_0\sin\mu_0 + \sin\vartheta_0\cos\varphi_0\sin\varphi_0$$
$$+ 2\cos\vartheta_0\sin^2\mu_0\}\Big] + \cdots .$$

Eqn. (96)

To evaluate HG, consider the right triangle HNG.

$$HG = \frac{NG}{\sin i_3}$$

$$= \frac{\frac{v}{c}(D_1'' + D_2'' - D_1' - D_2')\sin(\vartheta+\varphi)\ \operatorname{ctn}(\vartheta' + \sigma + i_2')}{\sin(\frac{\pi}{2} - \vartheta' - \sigma - i_2')}.$$

Simplifying, we obtain:

$$HG = \frac{\frac{v}{c}(D_1'' + D_2'' - D_1' - D_2')\sin(\vartheta+\varphi)}{\sin(\vartheta' + \sigma + i_2')}.$$

Substitute equations 5, 46 and 94.

$$HG = \frac{v}{c}\sin(\vartheta_0+\varphi_0)\Big[2(s_0'' - s_0')\frac{1}{\sin\vartheta_0} - 2\frac{v}{c}\frac{1}{\sin^2\vartheta_0}\{(s_0''-s_0')\sin(2\vartheta_0+\varphi_0)\cos\vartheta_0 + s_0''\cos\vartheta_0\sin\mu_0\}+\ldots\Big].$$

Substituting the value of C and D in equation 71, we obtain:

$$HG(1+\frac{v}{c}C + \frac{v^2}{c^2}D) = 2\frac{v}{c}(s_0''-s_0')\frac{\sin(\vartheta_0+\varphi_0)}{\sin\vartheta_0}$$

$$+ 2\frac{v^2}{c^2}\frac{\sin(\vartheta_0+\varphi_0)}{\sin^2\vartheta_0}\{-(s_0''-s_0')[2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0\sin\vartheta_0 + \sin(2\vartheta_0+\varphi_0)\cos\vartheta_0]$$

$$- s_0''\cos\vartheta_0\sin\mu_0\}+\ldots$$

Eqn. (97)

Now, in equation 90, substitute equation 96 for $LG/\frac{v}{c}$ and multiply. Then replace $HG(1+\frac{v}{c}C + \frac{v^2}{c^2}D)$ by equation 97. Arranging according to powers of v/c, we obtain:

$$N_3\lambda_0 = -\frac{v}{c}\Big[2(S_0''-S_0')\frac{1}{\sin\vartheta_0}\{\sin(2\vartheta_0+\varphi_0)\cos\vartheta_0-\sin(\vartheta_0+\varphi_0)\}+2S_0''\frac{\cos\vartheta_0}{\sin\vartheta_0}\sin\mu_0\Big]$$

$$-\frac{v^2}{c^2}\frac{1}{\sin^2\vartheta_0}\Big[-2(S_0''-S_0')\Big(\{\sin(2\vartheta_0+\varphi_0)\sin(\vartheta_0+\varphi_0)\cos\vartheta_0$$

$$-2\cos(\vartheta_0+\varphi_0)\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0-\sin(2\vartheta_0+\varphi_0)\sin(\vartheta_0+\varphi_0)\cos\vartheta_0$$

$$+2\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0$$

$$+2\sin(2\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\cos^2\vartheta_0\}$$

$$-\frac{\sin(2\vartheta_0+\varphi_0)}{\sin\mu_0}\{\tfrac{1}{4}\sin^2\vartheta_0\cos^2\vartheta_0\sin^2(\vartheta_0+\varphi_0)$$

$$+\tfrac{1}{4}\sin^2(2\vartheta_0+\varphi_0)\sin^4\vartheta_0-\tfrac{1}{4}\sin^2\vartheta_0\sin^2\varphi_0$$

$$-\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin^3\vartheta_0\cos\vartheta_0\}$$

$$+\sin\mu_0\sin(2\vartheta_0+\varphi_0)\sin^2\vartheta_0\Big)$$

$$-2S_0''\Big(\sin\vartheta_0\cos\vartheta_0\sin\varphi_0\cos\varphi_0-\tfrac{1}{4}\sin^2\vartheta_0\cos^2\vartheta_0\sin^2(\vartheta_0+\varphi_0)$$

$$-\tfrac{1}{4}\sin^2(2\vartheta_0+\varphi_0)\sin^4\vartheta_0+\tfrac{1}{4}\sin^2\vartheta_0\sin^2\varphi_0$$

$$+\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin^3\vartheta_0\cos\vartheta_0$$

$$+\sin\mu_0\{\sin(2\vartheta_0+\varphi_0)\cos^2\vartheta_0+2\cos(2\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0$$

$$+2\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\cos^2\vartheta_0\}$$

$$+\sin^2\mu_0\{2\cos^2\vartheta_0+\sin^2\vartheta_0\}\Big)\Big]+\cdots$$

Upon simplifications, this becomes:

$$N_3\lambda_0 = -\frac{v}{c}\Big[2(S_0''-S_0')\cos(2\vartheta_0+\varphi_0)+2S_0''\frac{\cos\vartheta_0}{\sin\vartheta_0}\sin\mu_0\Big]$$

$$-\frac{v^2}{c^2}\frac{1}{\sin^2\vartheta_0}\Big[-2(S_0''-S_0')\Big(2\cos(2\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\{\cos(\vartheta_0+\varphi_0)\sin\vartheta_0+\sin(2\vartheta_0+\varphi_0)\}$$

$$-\frac{\sin(2\vartheta_0+\varphi_0)}{\sin\mu_0}\{\tfrac{1}{4}\sin^2\vartheta_0\cos^2\vartheta_0[\sin^2(\vartheta_0+\varphi_0)-\sin^2(2\vartheta_0+\varphi_0)]\}$$

$$+\sin\mu_0\sin(2\vartheta_0+\varphi_0)\sin^2\vartheta_0\Big)$$

$$-2S_0''\Big(\sin\vartheta_0\cos\vartheta_0\{\sin\varphi_0\cos\varphi_0$$

$$+\tfrac{1}{2}\sin\vartheta_0\cos\vartheta_0[\sin^2(2\vartheta_0+\varphi_0)-\sin^2(\vartheta_0+\varphi_0)]\}$$

$$+\sin\mu_0\sin\vartheta_0\cos\vartheta_0\{3\cos(2\vartheta_0+\varphi_0)+2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0\}$$

$$+\sin^2\mu_0(1+\cos^2\vartheta_0)\Big)\Big]+\cdots$$

Eqn. (98)

TOTAL DIFFERENCE OF NUMBER OF WAVE LENGTHS IN THE TWO PATHS

To find the total difference of the number of wave lengths in the two paths, it is necessary to subtract equation 98 from equation 85. Let:

$$N_1 + N_2 - N_3 = N$$

Then:

$$\begin{aligned} N\lambda_0 = {} & 2(S_0'' - S_0') + \frac{v}{c}\left[-2(S_0''-S_0')\{\cos\varphi_0 - \cos(2\vartheta_0+\varphi_0)\}\right] \\ & + \frac{v^2}{c^2}\frac{1}{\sin^2\vartheta_0}\Big[2(S_0''-S_0')\{\tfrac{1}{2}\sin^2\vartheta_0 + \tfrac{3}{2}\sin^2\vartheta_0\cos^2\varphi_0 \\ & \qquad - 2\cos(2\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0[\cos(\vartheta_0+\varphi_0)\sin\vartheta + \sin(2\vartheta_0+\varphi_0)] \\ & \qquad + \frac{\sin(2\vartheta_0+\varphi_0)}{\sin\mu_0}[\tfrac{1}{4}\sin^2\vartheta\cos^2\vartheta_0(\sin^2(\vartheta_0+\varphi_0) - \sin^2(2\vartheta_0+\varphi_0))] \\ & \qquad - \sin\mu_0\sin(2\vartheta_0+\varphi_0)\sin^2\vartheta_0\} \\ & \quad + 2S_0''\{\sin\mu_0\sin\vartheta_0[2\cos(\vartheta_0+\varphi_0) + \cos\vartheta_0\cos\varphi_0] + \sin^2\vartheta_0\cos^2\varphi_0 \\ & \qquad - \sin\vartheta_0\cos\vartheta_0[\sin\varphi_0\cos\varphi_0 \\ & \qquad\qquad + \tfrac{1}{4}\sin\vartheta_0\cos\vartheta_0(\sin^2(2\vartheta_0+\varphi_0) - \sin^2(\vartheta_0+\varphi_0))] \\ & \qquad - \sin\mu_0\sin\vartheta_0\cos\vartheta_0[3\cos(2\vartheta_0+\varphi_0) + 2\cos(\vartheta_0+\varphi_0)\cos\vartheta_0]\} \\ & \qquad - S_0'\cos^2(2\vartheta_0+\varphi_0)\Big] + \cdots \end{aligned}$$

Eqn. (99)

The latter is the general equation of the difference of the number of wave lengths in the two paths, in terms of θ_0, ϕ_0, μ_0, S_0'', S_0', λ_0, v, and c.

APPLICATION OF THE FINAL RESULT TO THE WORK OF DAYTON C. MILLER

In the work of D. C. Miller[1] the two arms were made equal. Also, since the two arms were at right angles to each other θ_0 was equal to 45°. Substituting these conditions in equation 99, we obtain:

$$S_0'' - S_0' = 0$$

and

$$\sin\theta_0 = \cos\theta_0 = \frac{1}{\sqrt{2}}$$

Let S_0'' and S_0' be designated as S_0. Then:

$$N = \frac{4S_0}{\lambda_0}\frac{v^2}{c^2}\left[\sin\mu_0(\cos\varphi_0 + \sin\varphi_0) + \frac{15}{32}(\cos^2\varphi_0 - \sin^2\varphi_0) - \frac{7}{16}\sin\varphi_0\cos\varphi_0\right] + \cdots$$

In the latter equation let us substitute $\sin\omega_0$ for $v/c \sin\mu_0$. (cf. page 29). This gives:

$$N = \frac{4S_0}{\lambda_0}\frac{v}{c}\sin\omega_0(\cos\varphi_0 + \sin\varphi_0) + \frac{2S_0}{\lambda_0}\frac{v^2}{c^2}\left[\frac{15}{16}\cos 2\varphi_0 - \frac{7}{16}\sin 2\varphi_0\right] + \cdots$$

On page 227 of the article mentioned above, we find that:

$$\frac{2S_0}{\lambda_0} = 1.12 \times 10^8$$

Also, on page 238 of that same article, we find that ω_0 was equal to 4". Therefore:

$$\sin\omega_0 = 2 \times 10^{-5}$$

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1. Miller, D. C., Rev. Mod. Phys., 5, 203, (1933).

Let us substitute these values in the equation for N. This gives:

$$N = \frac{v}{c}\, 4.48\sqrt{2}\times 10^{3} \sin\left(\frac{\pi}{4}+\varphi_0\right) + \frac{v^2}{c^2}\left[1.05\times 10^{8}\cos 2\varphi_0 - .49\times 10^{8}\sin 2\varphi_0\right] + \cdots$$

Let:

$$A = 1.05\times 10^{8}$$

and

$$B = .49\times 10^{8}$$

Also, let:

$$A\cos 2\varphi_0 - B\sin 2\varphi_0 = C\sin(2\varphi_0 + \alpha_0)$$

$$= C\sin 2\varphi_0\cos\alpha_0 + C\cos 2\varphi_0\sin\alpha_0$$

Equating the coefficients of sin $2\phi_0$, we get:

$$B = -C\cos\alpha_0$$

Also, equating the coefficients of cos $2\phi_0$, we get:

$$A = C\sin\alpha_0$$

Dividing the latter by the former, we get:

$$\tan\alpha_0 = -\frac{A}{B}$$

$$= -2.14$$

Therefore:

$$\alpha_0 = 115^\circ 00'$$

Thus:

$$C = \frac{A}{\sin\alpha_0} = \frac{1.05\times 10^{8}}{.9063} = 1.16\times 10^{8}.$$

Substituting these values in the equation for N, we obtain:

$$N = \frac{v}{c}\, 6.34\times 10^{3}\sin\left(\frac{\pi}{4}+\varphi_0\right) + \frac{v^2}{c^2}\left[1.16\sin(2\varphi_0 + 115^\circ 00')\right]10^{8}.$$

Eqn. (100)

Finally, it will be desirable to plot the curves of the first and second order terms of equation 100 in order to compare them with the experimental curves obtained by D. C. Miller. So that the curves of the first and second order terms may be comparable, let us assume that v/c equals 10^{-4}.

Table of Values of First and Second Order

Terms for Various Values of ϕ_0

φ_0	$\frac{v}{c}\,6.34\times10^3\sin(\frac{\pi}{4}+\varphi_0)$	$\frac{v^2}{c^2}\,1.16\times10^8\sin(2\varphi_0+115^\circ00')$
0°	0.448	1.05
32°30'	.	0.00
45°	0.634 Max.	−0.49
77°30'	.	−1.16 Min.
90°	0.448	−1.05
122°30'	.	0.00
135°	0.000	0.49
167°30'	.	1.16 Max.
180°	−0.448	1.05
212°30'	.	0.00
225°	−0.634 Min.	−0.49
257°30'	.	−1.16 Min.
270°	−0.448	−1.05
302°30'	.	0.00
315°	0.000	0.49
347°30'	.	1.16 Max.
360°	0.448	1.05

Variation of Number of Fringes with ϕ_0

A. First Order Component B. Second Order Component

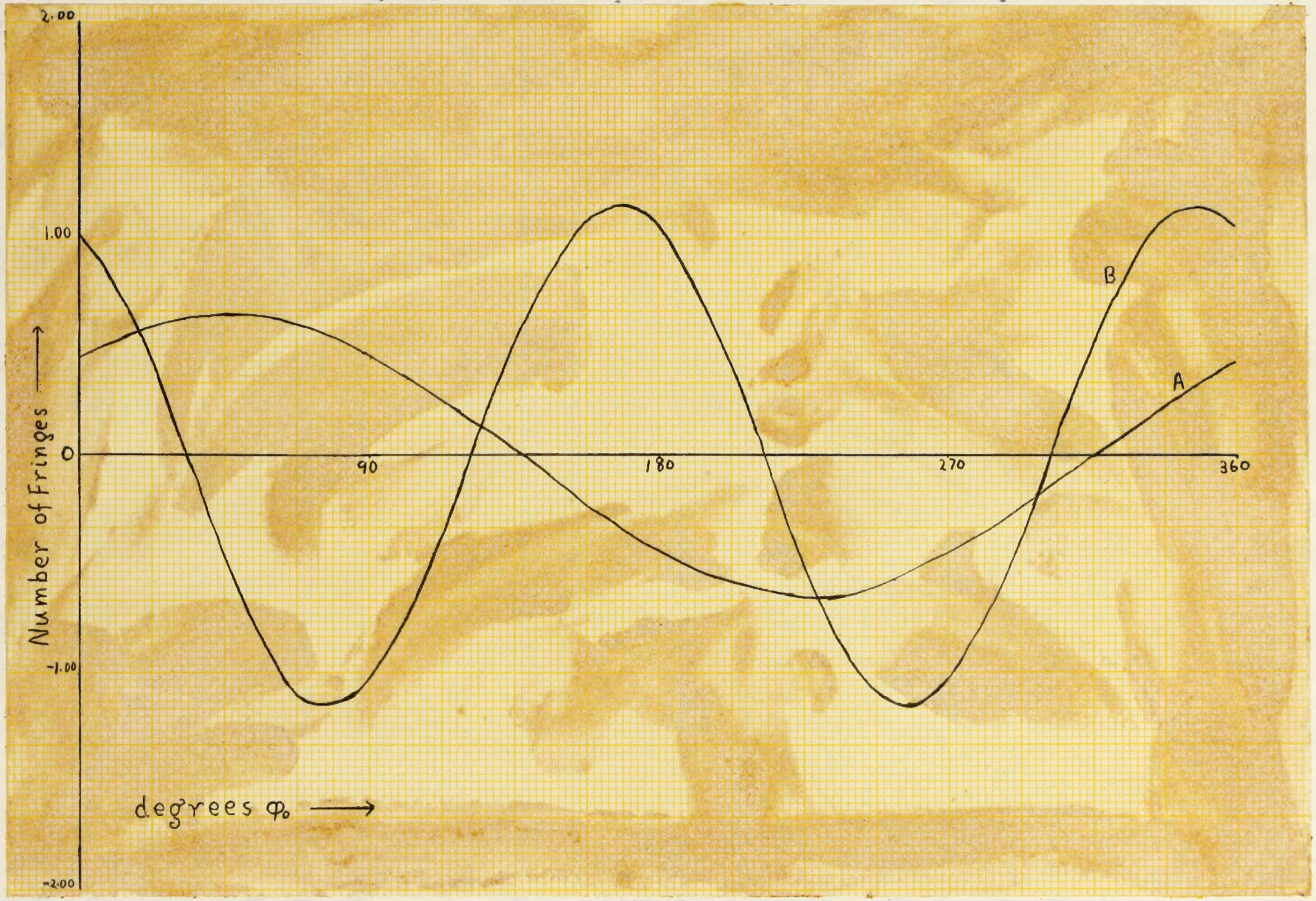

FIGURE VIII

CONCLUSION

On page 85 of this paper, we have a general equation of the difference of number of waves in the two paths traversed by the rays of light in a Michelson interferometer. According to this equation the actual fringe shift is a function of the effective difference in optical path, measured in terms of wave lengths along the various components of the paths from the time that the ray is first divided at the half silvered plate M until the rays have traveled their respective paths $(D_1'+D_2'-D_3')$ and $(D_1''+D_2''-D_3'')$. A telescope equally inclined to the two final rays must be focused at the point from which the two rays appear to be coming. In figure I that point is designated by T.

In deriving equation 85, no assumption was made concerning the relative lengths of the two arms S_0'' and S_0'. The only condition that need be imposed upon them is that $S_0''-S_0'$ shall not exceed the limits of "coherence". Otherwise, interference is impossible.

The angle between the plate and either arm, that is, θ_0 is entirely arbitrary and is merely determined by the conditions given in any particular arrangement of the apparatus. In most cases this angle is made to equal 45°. The value of θ_0 is very important in determining the maximum fringe shift.

The angle μ_0 is also a constant of any given interferometer. Its value is fixed so that the fringes observed in the telescope shall be of convenient width. Upon substituting the proper values in the equations of the law of reflection from a moving mirror (equations 21 and 22) and applying the

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In deriving equation 85, no assumption was made concerning the relative lengths of the two arms S_0'' and S_0'. The only condition that need be imposed upon them is that $S_0''-S_0'$ shall not exceed the limits of "coherence". Otherwise, interference is impossible.

The angle between the plate and either arm, that is, θ_0 is entirely arbitrary and is merely determined by the conditions given in any particular arrangement of the apparatus. In most cases this angle is made to equal 45°. The value of θ_0 is very important in determining the maximum fringe shift.

The angle ω_0 is also a constant of any given interferometer. Its value is fixed so that the fringes observed in the telescope shall be of convenient width. Upon substituting the proper values in the equations of the law of reflection from a moving mirror (equations 21 and 22) and applying the

Lorentz transformations it was found that these equations did not reduce to the simple law of reflection, namely that the angle of incidence is equal to the angle of reflection. This suggests that the angle of reflection depends upon the sidereal time. This conclusion is consistent with the results of E. Esclangon[1] and E. Cavallo[2].

In order to obtain a curve of the number of fringes plotted against ϕ_0, the constants of D. C. Miller's interferometer at Mount Wilson were substituted in equation 99. This led finally to equation 100. In this equation the first order term is periodic in a full turn of the instrument. Although the latter is a first order term, its magnitude is comparable with that of the second order term. The reason for this is that $\sin\omega_0$ which is a factor of the first order term is itself of the magnitude of v/c. Since, in substituting for $\sin\mu_0$ equation 99, that term dropped to the first order, it might argued that, had there been a third order term, the part of it that would have as its coefficient $\sin\mu_0$, would drop to the second term. However, since its magnitude would be comparable with that of the third order terms it is negligible to the order of approximation of this work.

In examining the first order term we see that when $\phi_0 = 45°$, the fringe shift due to this component is a maximum on one side of the zero mark in the telescope and for $\phi_0 = 225°$ it is a maximum on the other side.

The second order term of equation 100 is doubly periodic upon a single rotation of the interferometer. The maxima due to this component are at $\phi_0 = 167°30'$ and $\phi_0 = 347°30'$ on one side of the zero mark in the teles-

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1. Esclangon, E., Journal des Observateurs 11, 49, (1928)

2. Cavallo, E., Comptes Rendus, 198, 247, (1934)

cope and at $\phi_0 = 77^\circ 30'$ and $\phi_0 = 257^\circ 30'$ on the other side of the zero mark.

On page 227 of the article by D. C. Miller, there are graphs, obtained by a harmonic analyzer, of the experimental curves showing the readings for four successive sets of observations made on April 2, 1925. Comparing those curves with the theoretical curves given on page 89 of this paper, we see that the latter set of curves is in form similar to those obtained by experiment.

In conclusion, we may say, therefore, that if the equations governing the Michelson interferometer are set up from the point of view of an observer fixed in the ether and are transformed by means of the Lorentz transformation the results required by theory are consistent with those obtained by experiment. Until now, it has generally been assumed that the theory required a zero effect contrary to that obtained by D. C. Miller.

It is the authors pleasure to acknowledge his indebtedness for the very valuable suggestions and criticisms received from Dr. Royal M. Frye, under whose direction this problem has been carried out.

COMPREHENSIVE ABSTRACT

In this dissertation the general theory of the measurement of the ether drift by a Michelson interferometer is developed. The work is based upon three hypotheses which are widely used in physics. The first of these is that light is propagated as a wave disturbance according to Huygens' principle. The second hypothesis is that there exists a stationary ether through which light is propagated, and with respect to which the earth is moving with some relative velocity. The third hypothesis employed in this paper is one of the basic hypotheses of the theory of Relativity, namely that the velocity of light has a constant value relative to all inertial systems.

The interferometer contains a half-silvered mirror which partially transmits and partially reflects parallel light which is incident upon it. The transmitted and reflected rays are reflected at the end of their paths so as to return to the plate, where again one beam is reflected and the other is transmitted. Finally both rays are brought to a focus by a lens. The division and subsequent reuniting of a beam of light under these conditions brings about what are known as "interference fringes". If the hypotheses mentioned previously are correct, the position of these fringes should depend upon the orientation of the instrument with respect to the ether. The interference fringes themselves are due to the phase relationships existing between the wave fronts of the two reunited rays. These in turn are affected by the amount and direction of the motion of the interferometer during the interval of time that the beam of light is divided at the half-silvered plate and subsequently

brought to focus by the viewing telescope. In consequence of this, as the interferometer is rotated the interference fringe will show slight displacement unless the axis of rotation of the instrument is parallel to the direction of the absolute velocity of the earth.

In the investigation, the path of each beam is carefully traced, from the time that it impinges upon the plate, until it enters the lens placed in its final path.

Due to the motion of the earth all distances, including the wave lengths of the light, are subjected to a contraction. This latter is a necessary consequence of the third hypothesis mentioned previously. The equation governing this contraction is the familiar Lorentz transformation equation:

$$x_b - x_a = (x_{o_b} - x_{o_a})\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$$

where x_b and x_a are coordinates of a rod in a coordinate system fixed in the ether and x_{o_b} and x_{o_a} are the coordinates of a rod in a parallel moving coordinate system, $\underline{v}$ is the velocity of the moving system relative to the ether and **C** is the velocity of light. Angles and distances bearing the subscript zero will indicate those measured by the moving observer, all others will indicate these measured by the fixed observer.

Also, due to the motion of the earth, all angles undergo a change upon applying the Lorentz transformation. In the case of angles, two types must be dealt with. The first type includes angles of which one side is parallel to the direction of the motion. An equation characteristic of this type is:

$$\sin\varphi = \sin\varphi_o\left(1+\frac{1}{2}\frac{v^2}{c^2}\cos^2\varphi_o+\cdots\right).$$

where ϕ is an angle one side of which is parallel to the direction of drift.

The second type of angles consists of those of which neither side is parallel to the direction of the drift. An equation characteristic of this type is:

$$\sin\vartheta = \sin\vartheta_0\left[1 - \frac{1}{2}\frac{v^2}{c^2}\{\sin^2(\vartheta_0+\varphi_0) - \cos^2\varphi_0\} + \cdots\right].$$

where ϕ is the angle one side of which is parallel to the direction of the drift and θ is an angle adjacent to its other side.

In consequence of the change of angles, the mirror of the apparatus that is set normal to the arm to which it is fixed, will not be normal as viewed by the observer fixed in the ether. This discrepancy was evaluated and taken into consideration. Similarly, the angle introduced between the normal to the second mirror and the arm to which it is fixed undergoes a modification. Likewise, this effect is considered.

It is well known that in the case of a moving mirror the angle of incidence is not equal to the angle of reflection. The equations governing this change were found to be:

$$\sin i' = \sin i - 2\frac{v}{c}\sin\varphi\sin i\cos i + 2\frac{v^2}{c^2}\sin^2\varphi\sin i(2\cos^2 i - 1) + \cdots$$

and

$$\cos i' = \cos i + 2\frac{v}{c}\sin\varphi\sin^2 i - 4\frac{v^2}{c^2}\sin^2\varphi\sin^2 i\cos i + \cdots.$$

where i' is the angle of reflection, i the angle of incidence and ϕ is the angle between the direction of drift and the mirror. After substituting the proper values in the general equation, the Lorentz transformations were applied.

After determining the effect of the motion upon the angles involved in the path, the exact effect of the motion upon the length of each path is

where φ is an angle one side of which is parallel to the direction of drift.

The second type of angles consists of those of which neither side is parallel to the direction of the drift. An equation characteristic of this type is:

$$\sin\theta = \sin\theta_0\left[1 - \frac{1}{2}\frac{v^2}{c^2}\left(\sin^2(\theta_0+\varphi_0) - \cos^2\varphi_0\right) + \cdots\right]$$

where φ is the angle one side of which is parallel to the direction of the drift and θ is an angle adjacent to its other side.

In consequence of the change of angles, the mirror of the apparatus that is set normal to the arm to which it is fixed, will not be normal as viewed by the observer fixed in the ether. This discrepancy was evaluated and taken into consideration. Similarly, the angle introduced between the normal to the second mirror and the arm to which it is fixed undergoes a modification. Likewise, this effect is considered.

It is well known that in the case of a moving mirror the angle of incidence is not equal to the angle of reflection. The equations governing this change were found to be:

$$\sin i' = \sin i - 2\frac{v}{c}\sin\varphi\sin i\cos i + 2\frac{v^2}{c^2}\sin^2\varphi\sin i(2\cos^2 i - 1) + \cdots$$

and

$$\cos i' = \cos i + 2\frac{v}{c}\sin\varphi\sin^2 i - 4\frac{v^2}{c^2}\sin^2\varphi\sin^2 i\cos i + \cdots$$

where i' is the angle of reflection, i the angle of incidence and φ is the angle between the direction of drift and the mirror. After substituting the proper values in the general equation, the Lorentz transformations were applied.

After determining the effect of the motion upon the angles involved in the path, the exact effect of the motion upon the length of each path is

found.

Because of the motion, the wave length, λ' of the reflected light differs from λ, the wave length of the incident light according to the equation:

$$\frac{\lambda'}{\lambda} = \frac{\sin i'}{\sin i}$$

After substitution in this equation, the Lorentz transformations are applied. Then, by dividing each component of the two paths by the wave length of the light within that part of the path, we obtained the equations of the number of waves within that path. Having obtained the latter equations, the number of wave lengths in one complete path was subtracted from the number in the other complete path to give the total difference of number of wave lengths in the two paths. This latter equation was given in terms of: λ_0, the fundamental wave length; ϕ_0, the angle between the main arm of the interferometer and the direction of the motion; θ_0, the angle between the half-silvered plate and either arm; μ_0, an angle proportional to the angle between the normal to one of the mirrors and the arm to which it is fixed, the latter being introduced so that the fringe viewed in the telescope shall be of a convenient width; and the lengths of the two arms S_0'' and S_0'. The equation of the total difference in the number of fringes is entirely in terms of variables as measured by the moving observer.

Because of the great complexity of this problem it was found necessary to make approximations. Accordingly, all equations were developed only up to the second power of v/c. Powers of v/c greater than the second were neglected. This is a justifiable approximation, since the velocity of the earth in its orbit is thirty kilometers per second and the velocity of light is ten

thousand times as great. Hence, since v/c is of the order of 10^{-4}, $(v/c)^3$ is negligible.

Finally, to analyze the solution, the values of θ_0, S_0'', S_0', λ_0 and μ_0 used by D. C. Miller[1] were substituted. After assigning an arbitrary value to v/c, the first and second order terms were plotted and found to be of the form of the experimental curves obtained by D. C. Miller[2]. Thus, making the hypotheses mentioned in the beginning of this abstract, we find that, contrary to the usual opinion, it is possible to derive equations involving both the first and second order effects obtained by the very extensive work of D.C. Miller.

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1. Miller, D.C., Rev. Mod. Phys., 5, 203, (1933)

2. Ibid page 227

APPENDIX A

$$\sin\epsilon = \cos(\vartheta' + \sigma + i_2' + j_2')$$

Expanding, we obtain:

$$\sin\epsilon = \cos(\vartheta' + \sigma + i_2')\cos j_2' - \sin(\vartheta' + \sigma + i_2')\sin j_2'$$

Substitute equations 43, 45, 55 and 56.

$$\begin{aligned}
\sin\epsilon = \Big[&\cos\vartheta_0 - \tfrac{v}{c}\sin(2\vartheta_0+\varphi_0)\sin\vartheta_0 \\
&+ \tfrac{v^2}{c^2}\Big\{\tfrac{1}{2}\cos^2(2\vartheta_0+\varphi_0)\cos\vartheta_0 + \sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0 - \tfrac{1}{2}\cos^2(\vartheta_0+\varphi_0)\cos\vartheta_0 \\
&\quad - \tfrac{1}{2}\sin^2\varphi_0\cos\vartheta_0 - 2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin\vartheta_0\cos^2\vartheta_0 \\
&\quad - 4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\sin\vartheta_0\Big\}\Big]\Big[\sin\vartheta_0 + \tfrac{v}{c}\cos\vartheta_0\{\sin(2\vartheta_0+\varphi_0) + 2\sin\mu_0\} \\
&+ \tfrac{v^2}{c^2}\Big\{-\tfrac{1}{2}\sin^2(\vartheta_0+\varphi_0)\sin\vartheta_0 + \tfrac{1}{2}\sin\vartheta_0\cos^2\varphi_0 - \tfrac{1}{2}\sin\vartheta_0\sin^2\varphi_0 \\
&\quad + 4\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\sin\varphi_0 - 4\sin^2(\vartheta_0+\varphi_0)\sin\vartheta_0\cos^2\vartheta_0 \\
&\quad + 2\sin\vartheta_0\sin\varphi_0\sin\mu_0 + 2\cos\vartheta_0\cos\varphi_0\sin\mu_0 \\
&\quad - 8\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\sin\mu_0 - 2\sin\vartheta_0\sin^2\mu_0\Big\}\Big] \\
-\Big[&\sin\vartheta_0 + \tfrac{v}{c}\sin(2\vartheta_0+\varphi_0)\cos\vartheta_0 \\
&+ \tfrac{v^2}{c^2}\Big\{-\tfrac{1}{2}\sin^2(2\vartheta_0+\varphi_0)\sin\vartheta_0 + \tfrac{1}{2}\cos^2(\vartheta_0+\varphi_0)\sin\vartheta_0 - \tfrac{1}{2}\sin^2\varphi_0\sin\vartheta_0 \\
&\quad - 2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin^2\vartheta_0\cos\vartheta_0 + 4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\cos\vartheta_0\Big\}\Big] \\
\Big[&\cos\vartheta_0 - \tfrac{v}{c}\sin\vartheta_0\{\sin(2\vartheta_0+\varphi_0) + 2\sin\mu_0\} \\
&+ \tfrac{v^2}{c^2}\Big\{\tfrac{1}{2}\cos^2(\vartheta_0+\varphi_0)\cos\vartheta_0 + \sin\vartheta_0\sin\varphi_0\cos\varphi_0 - \tfrac{1}{2}\cos\vartheta_0 - 2\sin(\vartheta_0+\varphi_0)\sin\varphi_0 \\
&\quad + 4\sin(\vartheta_0+\varphi_0)\cos^2\vartheta_0\sin\varphi_0 + 4\sin^2(\vartheta_0+\varphi_0)\sin^2\vartheta_0\cos\vartheta_0 \\
&\quad - 2\sin^2(\vartheta_0+\varphi_0)\cos\vartheta_0 + 2\cos\vartheta_0\sin\varphi_0\sin\mu_0 - 2\sin\vartheta_0\cos\varphi_0\sin\mu_0 \\
&\quad + 4\sin(\vartheta_0+\varphi_0)\sin\mu_0 - 8\sin(\vartheta_0+\varphi_0)\cos^2\vartheta_0\sin\mu_0 - 2\cos\vartheta_0\sin^2\mu_0\Big\}\Big]
\end{aligned}$$

$$
\begin{aligned}
\sin\epsilon = {} & \cancel{\sin\vartheta_0\cos\vartheta_0} - \cancel{\sin\vartheta_0\cos\vartheta_0} \\
& + \frac{v}{c}\Big[\cancel{\cos^2\vartheta_0\sin(2\vartheta_0+\varphi_0)} + 2\cancel{\cos^2\vartheta_0}\sin\mu_0 - \cancel{\sin^2\vartheta_0\sin(2\vartheta_0+\varphi_0)} \\
& \quad - \cancel{\sin(2\vartheta_0+\varphi_0)\cos^2\vartheta_0} + \cancel{\sin(2\vartheta_0+\varphi_0)\sin^2\vartheta_0} + \cancel{2\sin^2\vartheta_0\sin\mu_0}\Big] \\
& + \frac{v^2}{c^2}\Big[-\overset{(1)}{\cancel{\tfrac{1}{2}\sin^2(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0}} + \tfrac{1}{2}\sin\vartheta_0\cos\vartheta_0\cos^2\varphi_0 \\
& \quad - \overset{(2)}{\cancel{\tfrac{1}{2}\sin\vartheta_0\sin^2\varphi_0\cos\vartheta_0}} + \overset{(3)}{\cancel{4\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos^2\vartheta_0\sin\varphi_0}} \\
& \quad - \overset{2\,(14)}{\cancel{4}}\sin^2(\vartheta_0+\varphi_0)\sin\vartheta_0\cos^{\overset{(4)}{\cancel{3}}}\vartheta_0 + \overset{(5)}{\cancel{2\sin\vartheta_0\cos\vartheta_0\sin\varphi_0\sin\mu_0}} \\
& \quad + 2\overset{(6)}{\cancel{\cos^2\vartheta_0}}\cos\varphi_0\sin\mu_0 - \overset{(7)}{\cancel{8\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos^2\vartheta_0\sin\mu_0}} \\
& \quad - \overset{(8)}{\cancel{2\sin\vartheta_0\cos\vartheta_0\sin^2\mu_0}} - \overset{(9)}{\cancel{\sin^2(2\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0}} \\
& \quad - \overset{(10)}{\cancel{2\sin(2\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\sin\mu_0}} + \overset{(11)}{\cancel{\tfrac{1}{2}\cos^2(2\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0}} \\
& \quad + \sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin^2\vartheta_0 - \overset{(1)}{\cancel{\tfrac{1}{2}\cos^2(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0}} \\
& \quad - \tfrac{1}{2}\sin^2\varphi_0\sin\vartheta_0\cos\vartheta_0 - \overset{(12)}{\cancel{2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin^2\vartheta_0\cos^2\vartheta_0}} \\
& \quad - 4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\overset{(13)}{\cancel{\sin^2\vartheta_0}} + \tfrac{1}{2}\sin^2(2\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0 \\
& \quad - \overset{(11)}{\cancel{\tfrac{1}{2}\cos^2(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0}} + \overset{(2)}{\cancel{\tfrac{1}{2}\sin^2\varphi_0\sin\vartheta_0\cos\vartheta_0}} \\
& \quad + \overset{(12)}{\cancel{2\sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0)\sin^2\vartheta_0\cos^2\vartheta_0}} - \overset{(13)}{\cancel{4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)\cos^2\vartheta_0}} \\
& \quad + \overset{(9)}{\cancel{\sin^2(2\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0}} + \overset{(10)}{\cancel{2\sin(2\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0\sin\mu_0}} \\
& \quad - \tfrac{1}{2}\cos^2(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0 - \sin^2\vartheta_0\sin\varphi_0\cos\varphi_0 \\
& \quad + \overset{(1)}{\cancel{\tfrac{1}{2}\sin\vartheta_0\cos\vartheta_0}} + 2\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\sin\varphi_0 \\
& \quad - \overset{(3)}{\cancel{4\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos^2\vartheta_0\sin\varphi_0}} - \overset{(4)}{\cancel{4\sin^2(\vartheta_0+\varphi_0)\sin^3\vartheta_0\cos\vartheta_0}} \\
& \quad + \overset{(14)}{\cancel{2\sin^2(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0}} - \overset{(5)}{\cancel{2\cos\vartheta_0\sin\varphi_0\sin\vartheta_0\sin\mu_0}} \\
& \quad + \overset{(6)}{\cancel{2\sin^2\vartheta_0\cos\varphi_0\sin\mu_0}} - 4\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\sin\mu_0 \\
& \quad + \overset{(7)}{\cancel{8\sin(\vartheta_0+\varphi_0)\sin\vartheta_0\cos^2\vartheta_0\sin\mu_0}} + \overset{(8)}{\cancel{2\sin\vartheta_0\cos\vartheta_0\sin^2\mu_0}}\Big] + \cdots
\end{aligned}
$$

After making the indicated cancellations, the above equation becomes:

$$\sin \epsilon = 2\frac{v}{c} \sin\mu_0$$
$$+\frac{v^2}{c^2}\Big[\tfrac{1}{2} \sin\vartheta_0 \cos\vartheta_0 \cos^2\varphi_0 - 2\sin^2(\vartheta_0+\varphi_0) \sin\vartheta_0 \cos\vartheta_0$$
$$+2\cos\varphi_0 \sin\mu_0 + \sin(\vartheta_0+\varphi_0)\cos(\vartheta_0+\varphi_0) \sin^2\vartheta_0$$
$$-\tfrac{1}{2}\sin^2\varphi_0 \sin\vartheta_0 \cos\vartheta_0 - 4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)$$
$$+\tfrac{1}{2}\sin^2(2\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0 - \tfrac{1}{2}\cos^2(\vartheta_0+\varphi_0)\sin\vartheta_0\cos\vartheta_0$$
$$-\sin^2\vartheta_0 \sin\varphi_0 \cos\varphi_0 + 2\sin(\vartheta_0+\varphi_0)\sin\vartheta_0 \sin\varphi_0$$
$$-4\sin(\vartheta_0+\varphi_0)\sin\vartheta_0 \sin\mu_0\Big] + \cdots$$

Upon further simplification, this reduces to:

$$\sin \epsilon = 2\frac{v}{c}\sin\mu_0 + \frac{v^2}{c^2}\Big[\tfrac{1}{2}\sin\vartheta_0\cos\vartheta_0\{\sin^2(2\vartheta_0+\varphi_0) - \sin^2(\vartheta_0+\varphi_0)\}$$
$$-4\sin(2\vartheta_0+\varphi_0)\cos(2\vartheta_0+\varphi_0)$$
$$+2\cos(2\vartheta_0+\varphi_0)\sin\mu_0\Big] + \cdots$$

Eqn. (58)

APPENDIX B

In deriving the equations for the length of the paths D_1' and D_2', it was tacitly assumed that the ray reflected from the plate M was incident upon the mirror M' at its center and that the ray reflected from the mirror M' was incident upon the plate M at its center. We shall now prove these facts.

In figure IX angles and distances corresponding to angles and distances in figure I have been designated by the same letters as in that figure. Let us assume that the ray ct_1' is incident upon the mirror M' at any arbitrary point C'. The angle ψ will still be defined as the angle between S' and the ray ct_1'. Also, i_2 will retain its definition as the angle of incidence of the ray at the mirror M'. Construct AC" perpendicular to the mirror M', then, in triangle AC'C" :

$$\cos i_2 = \frac{AC''}{AC'} = \frac{AC''}{ct_1'} .$$

In triangle AE"C" :

$$\text{Angle } AC''E'' = \sigma .$$

Therefore:

$$AC'' = (S' - EE'')\cos\sigma .$$

To find EE" construct EE' perpendicular to CC". Then, angle E'EE" equals σ. Also, in the right triangle CEE':

$$\text{Angle } CEE' = [\pi - (\vartheta + \vartheta' + \varphi + \sigma)]$$

Therefore:

$$EE' = vt_1' \cos[\pi - (\vartheta + \vartheta' + \varphi + \sigma)]$$

$$= -vt_1' \cos(\vartheta + \vartheta' + \varphi + \sigma)$$

Now, in triangle EE'E" :

$$EE'' = \frac{EE'}{\cos\sigma}$$

$$= - \frac{v t_1' \cos(\vartheta+\vartheta'+\varphi+\sigma)}{\cos\sigma} \; .$$

Substituting this expression for EE" in the equation for AC", we obtain:

$$AC'' = s'\cos\sigma + v t_1' \cos(\vartheta+\vartheta'+\varphi+\sigma)$$

In the first equation, replace AC" by this last expression, this gives:

$$c t_1' = \frac{s'\cos\sigma + v t_1' \cos(\vartheta+\vartheta'+\varphi+\sigma)}{\cos i_2} .$$

Solving for t_1', we get:

$$t_1' = \frac{s'\cos\sigma}{c\cos i_2 - v\cos(\vartheta+\vartheta'+\varphi+\sigma)}$$

Multiply both sides by c and replace ct_1' by D_1' . This gives:

$$D_1' = \frac{s'\cos\sigma}{\cos i_2 - \frac{v}{c}\cos(\vartheta+\vartheta'+\varphi+\sigma)}$$

From figure IX, we see that:

$$\psi = i_2 - \sigma , \quad \text{Also} \quad \psi = i_1' + \vartheta' - \frac{\pi}{2} .$$

Therefore,

$$i_2 = \vartheta' + \sigma + i_1' - \frac{\pi}{2}$$

and

$$\cos i_2 = \sin(\vartheta' + \sigma + i_1')$$

Expand and substitute the latter expression in the equation for D_1'. Also, divide numerator and denominator by cos σ and expand the v/c term. This yields:

$$D_1' = \frac{s'}{\sin(\vartheta'+i_1') + \cos(\vartheta'+i_1')\frac{\sin\sigma}{\cos\sigma} - \frac{v}{c}\cos(\vartheta+\vartheta'+\varphi) + \frac{v}{c}\sin(\vartheta+\vartheta'+\varphi)\frac{\sin\sigma}{\cos\sigma}}$$

In substituting the equations 15, 16 and 32, the second and the last terms of the denominator have as their coefficients powers of v/c greater than the second therefore, they will be neglected. What remains in the equation is the same as the expression for D_2' found on page 56 of the main body of this thesis.

Thus, since the two expressions for D_2' are the same, it must follow that the points C' and C are coincident and the ray actually strikes at the center of the mirror M'.

Having proved that the ray reflected from the plate M is incident upon the center of the mirror M', let us represent the ray reflected from M' by the line CH', where H' represents any point on the plate M. Let K represent the position of the plate M at the time t_2' and J the point of intersection of the ray ct_2' and the line AH.

From the figure IX, we see that:

$$ct_2' = D_2' = CJ + JH'$$

Now, in the triangle CJK :

$$\frac{\sin(\vartheta+\vartheta'+\varphi)}{CJ} = \frac{\sin[\pi-(\vartheta+\vartheta'+\varphi+\sigma+i_2')]}{s'} = \frac{\sin(i_2'+\sigma)}{KJ}$$

Therefore:

$$CJ = \frac{s'\sin(\vartheta+\vartheta'+\varphi)}{\sin(\vartheta+\vartheta'+\varphi+\sigma+i_2')}$$

and:

$$KJ = \frac{s'\sin(i_2'+\sigma)}{\sin(\vartheta+\vartheta'+\varphi+\sigma+i_2')}$$

From the figure IX, we observe that:

$$JH = vt_2' - KJ = vt_2' - \frac{s'\sin(i_2'+\sigma)}{\sin(\vartheta+\vartheta'+\varphi+\sigma+i_2')}$$

In the triangle JHH' we have, by the sine law:

$$JH' = \frac{JH \sin(\theta+\varphi)}{\sin(\theta'+\sigma+i_2')}$$

Substituting the previous equation for JH ,we obtain:

$$JH' = \frac{v t_2' \sin(\theta+\varphi)}{\sin(\theta'+\sigma+i_2')} - \frac{s' \sin(i_2'+\sigma)\sin(\theta+\varphi)}{\sin(\theta+\theta'+\varphi+\sigma+i_2')\sin(\theta'+\sigma+i_2')}$$

Now, in the equation for ct_2' substitute the equations for CJ and JH'. This gives:

$$ct_2' = \frac{s'\sin(\theta+\theta'+\varphi)}{\sin(\theta+\theta'+\varphi+\sigma+i_2')} + \frac{v t_2' \sin(\theta+\varphi)}{\sin(\theta'+\sigma+i_2')} - \frac{s' \sin(i_2'+\sigma)\sin(\theta+\varphi)}{\sin(\theta+\theta'+\varphi+\sigma+i_2')\sin(\theta'+\sigma+i_2')}.$$

Solving for t_2', we get:

$$t_2' = \frac{s'}{\sin(\theta+\theta'+\varphi+\sigma+i_2')}\left[\frac{\sin(\theta+\theta'+\varphi)\sin(\theta'+\sigma+i_2') - \sin(i_2'+\sigma)\sin(\theta+\varphi)}{c\sin(\theta'+\sigma+i_2') - v\sin(\theta+\varphi)}\right].$$

Multiply both sides of the equation by c and divide the numerator and the denominator of the right hand member by $\sin(\theta'+\sigma+i_2')$. This gives:

$$D_2' = \frac{s'}{\sin(\theta+\theta'+\varphi+\sigma+i_2')}\left[\frac{\sin(\theta+\theta'+\varphi) - \dfrac{\sin(i_2'+\sigma)\sin(\theta+\varphi)}{\sin(\theta'+\sigma+i_2')}}{1 - \dfrac{v}{c}\dfrac{\sin(\theta+\varphi)}{\sin(\theta'+\sigma+i_2')}}\right]$$

Expanding the denominator, we get:

$$D_2' = \frac{s'}{\sin(\theta+\theta'+\varphi+\sigma+i_2')}\left[\sin(\theta+\theta'+\varphi) - \frac{\sin(i_2'+\sigma)\sin(\theta+\varphi)}{\sin(\theta'+\sigma+i_2')}\right]$$
$$\left[1 + \frac{v}{c}\frac{\sin(\theta+\varphi)}{\sin(\theta'+\sigma+i_2')} + \frac{v^2}{c^2}\frac{\sin^2(\theta+\varphi)}{\sin^2(\theta'+\sigma+i_2')}\right].$$

Multiplying the two brackets and observing that $\sin(i_2'+\sigma)$ is of the order of v/c, we get:

$$D_2' = \frac{s'}{\sin(\theta+\theta'+\varphi+\sigma+i_2')}\Big[\sin(\theta+\theta'+\varphi) + \frac{\sin(\theta+\varphi)}{\sin(\theta'+\sigma+i_2)}\left\{\frac{v}{c}\sin(\theta+\theta'+\varphi) - \sin(i_2'+\sigma)\right\}$$
$$+ \frac{\sin^2(\theta+\varphi)}{\sin^2(\theta'+\sigma+i_2')}\left\{\frac{v^2}{c^2}\sin(\theta+\theta'+\varphi) - \frac{v}{c}\sin(i_2'+\sigma)\right\} + \cdots\Big]$$

Expanding $\sin(i_2' + \sigma)$ and substituting equations 15, 16, 39 and 40; also, replacing $\sin(\theta + \theta' + \phi)$ by equation 7, we find that the two braces reduce to zero. Therefore:

$$D_2' = s' \frac{\sin(\vartheta + \vartheta' + \varphi)}{\sin(\vartheta + \vartheta' + \varphi + \sigma + i_2')}$$

It will be observed that this latter expression is the same as that for D_2' at the top of page 58.

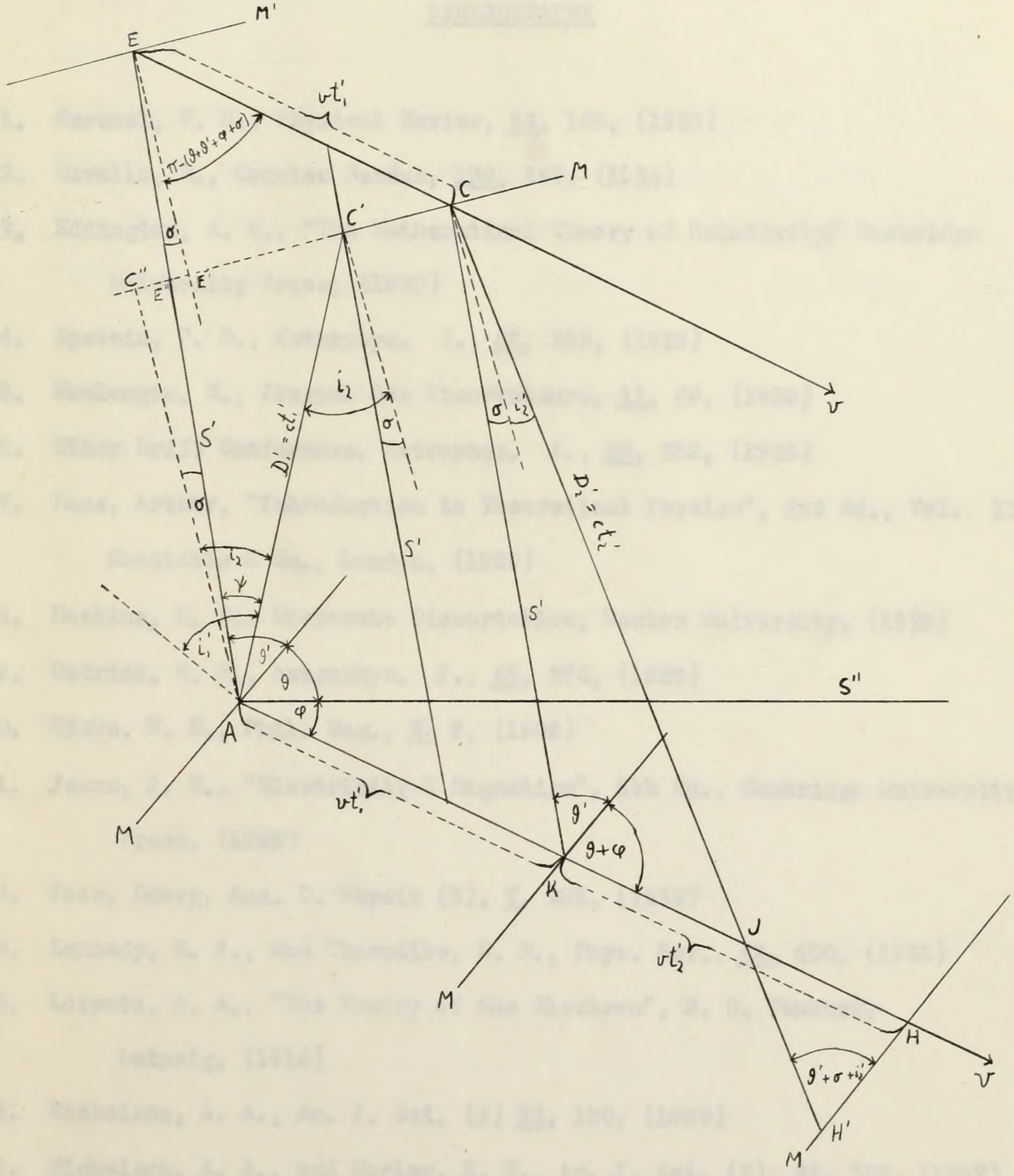
Thus, since the two expressions for D_2' are the same, it must follow that the points H' and H are coincident. Hence, the ray actually strikes at the center of the plate M.

Expanding $\sin(\gamma_1' + r)$ and substituting equations 15, 26, 35 and 60; also, replacing $\sin(\theta + \theta' + \gamma_1)$ by equation 7, we find that the two brackets reduce to zero. Therefore:

$$D_1' = S' \frac{\sin(\gamma - \gamma' + \theta)}{\sin(\gamma + \gamma' + \theta + \theta' + \gamma_1)}$$

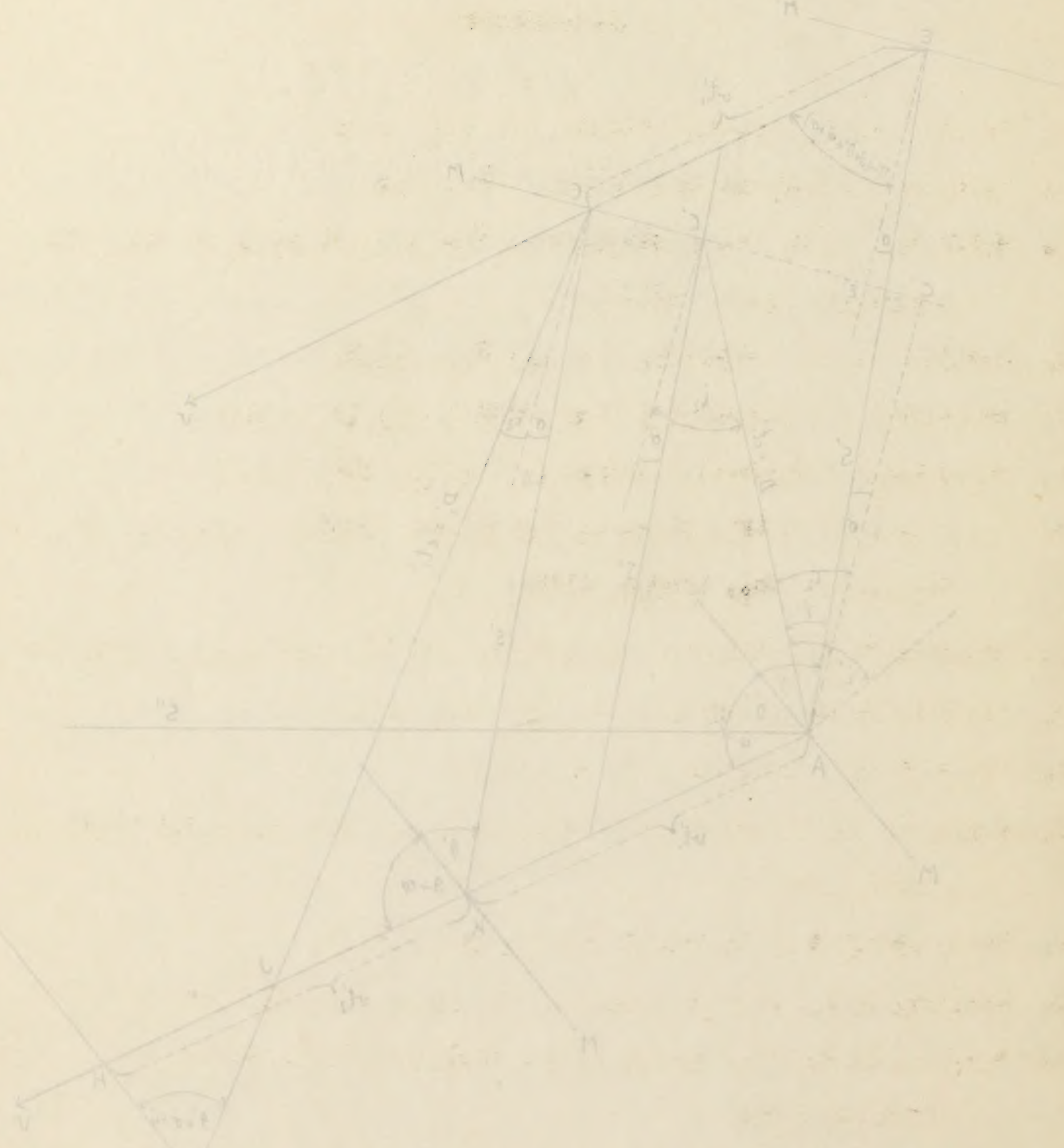
It will be observed that this latter expression is the same as that for D_1' at the top of page 58.

Thus, since the two expressions for D_1' are the same, it must follow that the points D' and H are coincident. Hence, the ray actually strikes at the center of the plate H.



E
M'
vt_1'
$\pi-(\vartheta+\vartheta'+\varphi+\sigma)$
M
C
C'
σ
C''
E
i_1
σ
σ
i_2
S'
$D_1=ct_1$
σ
$D_1'=ct_2'$
v
i_2
ψ
S'
i_1'
ϑ'
ϑ
φ
S''
A
S'
M
vt_1'
ϑ'
$\vartheta+\varphi$
K
J
vt_2'
M
H
$\vartheta'+\sigma+i_2'$
v
H'
M

Figure IX



BIBLIOGRAPHY

1. Cartmel, W. B., Physical Review, 53, 108, (1938)

2. Cavallo, E., Comptes Rendus, 198, 247, (1934)

3. Eddington, A. S., "The Mathematical Theory of Ralativity," Cambridge University Press, (1930)

4. Epstein, P. S., Astrophys. J., 68, 383, (1928)

5. Esclangon, E., Journal des Observateurs, 11, 49, (1928)

6. Ether Drift Conference, Astrophys. J., 68, 352, (1928)

7. Haas, Arthur, "Introduction to Theoretical Physics", 2nd Ed., Vol. II Constable & Co., London, (1929)

8. Haskins, E. E., Doctorate Dissertation, Boston University, (1938)

9. Hedrick, E. R., Astrophys. J., 68, 374, (1928)

10. Hicks, W. M., Phil. Mag., 3, 9, (1902)

11. Jeans, J. H., "Electricity & Magnetism", 5th Ed., Cambridge University Press, (1925)

12. Joos, Georg, Ann. D. Physik (5), 7, 385, (1930)

13. Kennedy, R. J., and Thorndike, E. M., Phys. Rev., 42, 400, (1932)

14. Lorentz, H. A., "The Theory of the Electron", B. G. Teubner, Leipzig, (1916)

15. Michelson, A. A., Am. J. Sci. (3) 22, 120, (1880)

16. Michelson, A. A., and Morley, E. W., Am. J. Sci. (3), 34, 333, (1887)

17. Miller, D. C., Science, 61, 617, (1925)

18. Miller, D. C., Nature, 117, 890, (1926)

BIBLIOGRAPHY (Cont.)

19. Miller, D. C., Science, 63, 433, (1926)

20. Miller, D. C., Science, 77, 587, (1933)

21. Miller, D. C., Rev. Mod. Phys., 5, 203, (1933)

22. Morley, E. W. and Miller, D. D., Phil. Mag. (6), 9, 680, (1905)

23. Piccard, A. and Stahel, E., Comptes Rendus, 183, 420, (1926)

24. Piccard, A. and Stahel, E., Comptes Rendus, 185, 1198, (1927)

25. Richtmyer, F. K., "Introduction to Modern Physics", McGraw Hill Book Co. (1934)

26. Righi, A., Comptes Rendus, 168, 837, (1919)

27. Righi, A. Comptes Rendus, 170, 499 and 1550, (1920)

AUTOBIOGRAPHY

The author was born July 3, 1913, in Boston, Massachusetts, the son of David and Celia Lisman. He was graduated from the Wellington Grammar School in Cambridge in 1926, and from the Dorchester High School in 1930.

In June 1934 he received the degree of Bachelor of Science in Education from Boston University; his major fields of study having been education, physics, mathematics and chemistry. In June 1935 he received the degree of Master of Arts from Boston University. During that year, he was an assistant in the Boston University physics department. In the following four academic years, he took part time work at Harvard University and Boston University leading to the degree of Doctor of Philosophy.

During all the years that he has been attending the University with the exception of the year 1934-35, he has been teaching in private schools in Lowell and in Boston. He was married to Rachel Lewit on April 3, 1938.